1. Gegeben sei die Matrix

$$A = \begin{pmatrix} -3 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Gesucht ist die Fundamentallösungsmatrix des linearen Gleichungssystems $\dot{x} = Ax$.

$$\begin{vmatrix} -3-\lambda & 1 & 1 \\ 0 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda^2(3+\lambda) \rightarrow \lambda_1 = \lambda_2 = 0 \quad (\text{Alg. Vfh. } 2)$$
$$\lambda_3 = -3$$

$\underline{\lambda_1 = 0}$:

$$\begin{matrix} -3 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \rightsquigarrow \begin{matrix} -3 & 1 & 1 \\ 0 & 1 & 0 \end{matrix} \rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

(Geom. Vfh. 1)

$\underline{\lambda_2 = 0}$:

$$\left(\begin{matrix} -3 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \,\middle|\, \begin{matrix} 1 \\ 0 \\ 3 \end{matrix}\right) \rightsquigarrow \left(\begin{matrix} -3 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \,\middle|\, \begin{matrix} 1 \\ 3 \\ 0 \end{matrix}\right) \rightarrow \tilde{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

$\underline{\lambda_3 = -3}$:

$$\begin{matrix} 0 & 1 & 1 \\ 0 & +3 & 0 \\ 0 & 1 & 3 \end{matrix} \rightarrow \begin{matrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \rightarrow v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + c_2 \left(\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right) + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-3t}$$

$$\rightarrow \underline{\underline{\Phi(t) = \begin{pmatrix} 1 & 1+t & e^{-3t} \\ 0 & 3 & 0 \\ 3 & 1+3t & 0 \end{pmatrix}}}$$

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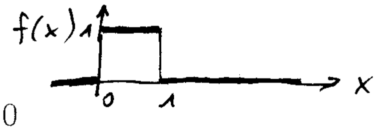
2. Lösen Sie mit Hilfe der Laplace Transformation



$$y'' + y = f(x) \qquad y(0) = 0, y'(0) = 0$$

$= H(x) - H(x-1)$

wobei $f(x) = 1$, wenn $0 < x < 1$ und sonst 0 ist.

$$\rightarrow s^2 Y + Y = \frac{1}{s}(1 - e^{-s}) \rightarrow Y = \underbrace{\frac{1}{s(s^2+1)}}_{G(s)} (1 - e^{-s})$$

$$G(s) = \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs + C}{s^2+1} \rightarrow 1 = As^2 + A + Bs^2 + Cs \Rightarrow \begin{matrix} A = 1 \\ B = -1 \\ C = 0 \end{matrix}$$

$$= \frac{1}{s} - \frac{s}{s^2+1} \Rightarrow g(x) = \underline{\underline{1 - \cos x}}$$

$$\rightarrow y(x) = (1 - \cos x) H(x) - [1 - \cos(x-1)] H(x-1) = \begin{cases} 1 - \cos x & 0 \le x \le 1 \\ -\cos x + \cos(x-1) & x > 1 \\ 0 & x < 0 \end{cases}$$

3. Lösen Sie

$$z_{tt} = 9z_{xx}$$

mit

$$z(0,t) = z(\pi,t) = 0$$
$$z(x,0) = \sin 2x \quad \text{in } (0,\pi)$$
$$z_t(x,0) = \frac{1}{6}\sin 6x \quad \text{in } (0,\pi)$$

$$z(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos 3nt + B_n \sin 3nt\right)\sin nx$$

$$z(x,0) = \sum_{n=1}^{\infty} A_n \sin nx \overset{!}{=} \sin 2x \rightarrow \frac{A_2 = 1}{A_n = 0 \text{ sonst}}$$

$$z_t(x,t) = \sum_{n=1}^{\infty}\left(-3nA_n \sin 3nt + 3nB_n \cos 3nt\right)\sin nx$$

$$z_t(x,0) = \sum_{n=1}^{\infty} 3nB_n \sin nx \overset{!}{=} \frac{1}{6}\sin 6x \rightarrow \frac{B_6 = \frac{1}{108}}{B_n = 0 \text{ sonst.}}$$

$$n=6: \quad 18B_6 = \frac{1}{6}$$

$$\Rightarrow z(x,t) = \left(\cos 6t \sin 2x + \frac{1}{108}\sin 18t \sin 6x\right)$$

4. Lösen Sie

$$x^2y'' - 5xy' + 9y = x^3$$

(Euler-Dgl.)

$$y = u \qquad x = e^t$$
$$y' = \dot{u}\cdot\frac{1}{x} \qquad \ln|x| = t$$
$$y'' = (\ddot{u} - \dot{u})\frac{1}{x^2}$$

$$\ddot{u} - \dot{u} - 5\dot{u} + 9u = e^{3t} \rightarrow \lambda^2 - 6\lambda + 9 = 0 \Rightarrow \lambda_{1,2} = 3$$

$$u_h(t) = c_1 e^{3t} + c_2 t e^{3t}$$

$$u_p(t) = At^2 e^{3t}$$
$$u_p'(t) = A(2t + 3t^2)e^{3t}$$
$$u_p''(t) = A(2 + 6t + 6t + 9t^2)e^{3t}$$

$$A\left[2 + \cancel{12t} + \cancel{9t^2} - \cancel{12t} - \cancel{18t^2} + \cancel{9t^2}\right] = 1 \Rightarrow A = \frac{1}{2}$$

$$u = c_1 e^{3t} + c_2 t e^{3t} + \frac{1}{2}t^2 e^{3t}$$

$$\Rightarrow y(x) = c_1 x^3 + c_2 x^3 \ln|x| + \frac{1}{2}x^3 \ln^2|x|$$