Mathematik 2 C **SS 2006**

Lösung
zur 3. Aufgabe

zu 1. (a)

$$z(1,1) = e^1 - 1 \doteq \underline{\underline{1.7183}}$$

(b)

$$\nabla z(x,y) = \begin{pmatrix} 2xye^{x^2y} - y^2 \\ x^2e^{x^2y} - 2xy \end{pmatrix} \quad \Rightarrow \quad \nabla z(1,1) = \begin{pmatrix} 2e^1 - 1 \\ e^1 - 2 \end{pmatrix} \doteq \underline{\underline{\begin{pmatrix} 4.4366 \\ 0.7183 \end{pmatrix}}}$$

(c)

$$k_{\max} = \|\nabla z(1,1)\| \doteq \sqrt{4.4366^2 + 0.7183^2} \doteq \underline{\underline{4.494}}$$

(d)

$$\vec{a} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \Rightarrow \quad k_{\vec{a}} \doteq \frac{1}{\sqrt{5}}\left\langle \begin{pmatrix} 4.4366 \\ 0.7183 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\rangle \doteq \underline{\underline{-1.3416}}$$

zu 2. Die Ecken ergeben sich durch Gleichsetzen der Kurvengleichungen:

$$y \;=\; 2 = \frac{x^2}{2} \quad \Rightarrow \quad x = \pm 2 \quad \Rightarrow \quad y = 2 \quad \Rightarrow \quad P_1 = (-2,2),\; P_2 = (2,2)$$

- als Kurvenintegral:
  Obere Kurve: $y(x) = 2 \quad \Rightarrow \quad \frac{dy}{dx} = 0 \quad \Rightarrow \quad dy = 0$

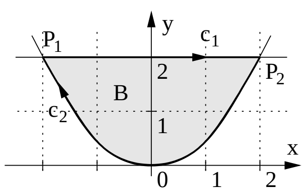
$$I_1 = \int_{c_1} dx + (y^2 - \sin x + 1)\, dy = \int_{x=-2}^{2} dx + 0 = \big[x\big]_{-2}^{2} = 4$$

Untere Kurve: $y(x) = \frac{x^2}{2} \quad \Rightarrow \quad \frac{dy}{dx} = x \quad \Rightarrow \quad dy = x\,dx$

$$I_2 = \int_{c_2} dx + (y^2 - \sin x + 1)\, dy = \int_{2}^{-2} dx + \left(\frac{x^4}{4} - \sin x + 1\right) x\, dx$$

$$= \int_{2}^{-2} \left(1 + \frac{x^5}{4} - x\sin x + x\right) dx = \left[x + \frac{x^6}{24} + x\cos x - \sin x + \frac{x^2}{2}\right]_{2}^{-2} \doteq -0.5168$$

$$\Rightarrow \quad \oint_c dx + (y^2 - \sin x + 1)\, dy = I_1 + I_2 \doteq \underline{\underline{3.4832}}$$



- mittels Green'scher Formel:

$$\oint_c \underbrace{1}_{f(x,y)} \, dx + \underbrace{(y^2 - \sin x + 1)}_{g(x,y)} \, dy \;=\; \iint_B \Big(\underbrace{-\cos(x)}_{\frac{\partial g}{\partial x}} - \underbrace{0}_{\frac{\partial f}{\partial y}} \Big) \, dy \, dx$$

$$= \int\limits_{x=-2}^{2} \int\limits_{y=\frac{x^2}{2}}^{2} -\cos(x) dy dx = \int\limits_{x=-2}^{2} -\cos(x) \, [y]_{\frac{x^2}{2}}^{2} \, dx = \int\limits_{x=-2}^{2} -\cos(x) \left(2 - \frac{x^2}{2}\right) dx$$

Nebenrechnung:

$$\int \left(\frac{x^2}{2} - 2\right) \cos(x) \, dx \;= \ldots = \; \left(\frac{x^2}{2} - 2\right) \sin(x) + x\cos(x) - \sin(x) + C$$

Einsetzen ins Integral:

$$\Rightarrow \quad \int\limits_{-2}^{2} \left(\frac{x^2}{2} - 2\right) \cos(x) \, dx \;=\; \left[\left(\frac{x^2}{2} - 3\right) \sin(x) + x\cos(x)\right]_{-2}^{2} \;\doteq\; -\underline{\underline{3.4832}}$$

Das Vorzeichen ist hier noch umzudrehen, da die Fläche im Bsp. im mathematischen Sinn „rückwärts“ umrundet wird!

zu 3.

$$\frac{\partial V_1}{\partial y} \;=\; 2xe^y + 2xye^y \;=\; 2x(1+y)e^y \;=\; \frac{\partial V_2}{\partial x} \quad \checkmark$$

$\Rightarrow \quad \vec{V}$ ist durch Gradientenbildung aus $\Phi(x,y)$ entstanden.

$$\overbrace{2x\big(ye^y - 6x\big)}^{\vec{V}_1} = \frac{\partial \Phi}{\partial x} \quad \Rightarrow \quad \Phi(x,y) = \int 2x\big(ye^y - 6x\big) \, dx = x^2ye^y - 4x^3 + \varphi(y)$$

$$\overset{\substack{\text{Abl.}\\ \text{nach } y}}{\Rightarrow} \quad \frac{\partial \Phi}{\partial y} = x^2e^y + x^2ye^y + \frac{d\varphi(y)}{dy} \overset{!}{=} \overbrace{x^2(1+y)e^y + \sqrt{y}}^{\vec{V}_2} \quad \Rightarrow \quad \frac{d\varphi(y)}{dy} = \sqrt{y}$$

$$\Rightarrow \quad \varphi(y) = \int \sqrt{y} \, dy = \tfrac{2}{3}y\sqrt{y} + C \quad \Rightarrow \quad \Phi(x,y) = x^2ye^y - 4x^3 + \tfrac{2}{3}y\sqrt{y} + C$$

$$\Phi(1,0) = 0 - 4 + 0 + C \overset{!}{=} -1 \;\Rightarrow\; C = 3 \;\Rightarrow\; \underline{\underline{\Phi(x,y) = x^2ye^y - 4x^3 + \tfrac{2}{3}y\sqrt{y} + 3}}$$

zu 4.

$$\operatorname{div} \vec{P}(x,y,z) = \frac{\partial P_1}{\partial x} + \frac{\partial P_2}{\partial y} + \frac{\partial P_3}{\partial z} = 2z + 2x + 2y = \underline{\underline{2(x+y+z)}}$$

$$\operatorname{rot} \vec{P}(x,y,z) = \begin{pmatrix} \frac{\partial P_3}{\partial y} - \frac{\partial P_2}{\partial z} \\ \frac{\partial P_1}{\partial z} - \frac{\partial P_3}{\partial x} \\ \frac{\partial P_2}{\partial x} - \frac{\partial P_1}{\partial y} \end{pmatrix} = \begin{pmatrix} 2z & -\, 2z \\ (2x - 2z) & -\, 2x \\ 2y & -\, 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 \\ -2z \\ 2y \end{pmatrix}}}$$