Mathematik 2 B WB — SS 2006

Musterlösung zum 2. Aufgabenblatt

zu 1.) (a)

$$\left.\begin{array}{r} p=-3<0 \\ q=8>0 \end{array}\right\} \quad \Rightarrow \quad \text{instabil}$$

$$p^2-4q=-23<0 \quad \Rightarrow \quad \text{Strudelpunkt}$$

(b)

$$\left.\begin{array}{l} p=-120<0 \\ q=3600>0 \end{array}\right\} \quad \Rightarrow \quad \text{instabil}$$

$$p^2-4q=0 \quad \Rightarrow \quad \text{uneigentlicher Knoten}$$

(c)

$$\left.\begin{array}{l} p=-\sqrt{3}+\sqrt{5} \doteq 0.504>0 \\ q=-\sqrt{15}+2 \doteq -1.873<0 \end{array}\right\} \quad \Rightarrow \quad \text{instabil, Sattelpunkt}$$

zu 2.) Ruhelagen:

$$x(x+y-1)=0 \quad \Rightarrow \quad x=0 \quad \text{oder} \quad y=1-x$$

$$y(y-x+3)=0 \quad \Rightarrow \quad y=0 \quad \text{oder} \quad y=x-3$$

$$\Rightarrow \quad P_1=(0,0) \qquad P_2=(0,-3) \qquad P_3=(1,0) \qquad P_4=(2,-1)$$

zu P_1:

$$\begin{array}{l} \dot{x}=x^2+xy-x \\ \dot{y}=y^2-xy+3y \end{array} \quad \overset{\text{lin.}}{\rightsquigarrow} \quad \begin{array}{l} \dot{x}=-x \\ \dot{y}=3y \end{array} \qquad \left.\begin{array}{l} p=-2 \\ q=-3 \end{array}\right\} \text{Sattelpunkt, instabil}$$

zu P_2: Substitution $x:=\bar{x},\ y:=\bar{y}-3$

$$\begin{array}{l} \dot{\bar{x}}=\bar{x}^2+\bar{x}\bar{y}-4\bar{x} \\ \dot{\bar{y}}=\bar{y}^2-3\bar{y}-\bar{x}\bar{y}+3\bar{x} \end{array} \quad \overset{\text{lin.}}{\rightsquigarrow} \quad \begin{array}{l} \dot{\bar{x}}=-4\bar{x} \\ \dot{\bar{y}}=3\bar{x}-3\bar{y} \end{array} \qquad \left.\begin{array}{l} p=7 \\ q=12 \\ p^2-4q=1 \end{array}\right\} \text{stabiler Kn.}$$

zu P_3: Substitution $x:=\bar{x}+1,\ y:=\bar{y}$

$$\begin{array}{l} \dot{\bar{x}}=\bar{x}^2+\bar{x}+\bar{x}\bar{y}+\bar{y} \\ \dot{\bar{y}}=\bar{y}^2-\bar{x}\bar{y}+2\bar{y} \end{array} \quad \overset{\text{lin.}}{\rightsquigarrow} \quad \begin{array}{l} \dot{\bar{x}}=\bar{x}+\bar{y} \\ \dot{\bar{y}}=2\bar{y} \end{array} \qquad \left.\begin{array}{l} p=-3 \\ q=2 \\ p^2-4q=1 \end{array}\right\} \text{instab. Knoten}$$

zu P_4: Substitution $x:=\bar{x}+2,\ y:=\bar{y}-1$

$$\begin{array}{l} \dot{\bar{x}}=\bar{x}^2+2\bar{x}+\bar{x}\bar{y}+2\bar{y} \\ \dot{\bar{y}}=\bar{y}^2-\bar{y}-\bar{x}\bar{y}+\bar{x} \end{array} \quad \overset{\text{lin.}}{\rightsquigarrow} \quad \begin{array}{l} \dot{\bar{x}}=2\bar{x}+2\bar{y} \\ \dot{\bar{y}}=\bar{x}-\bar{y} \end{array} \qquad \left.\begin{array}{l} p=-1 \\ q=-4 \end{array}\right\} \text{Sattel, inst.}$$

zu 3.) Gleichgewichtslagen:

$$f(x) = \frac{x^2-1}{x^2+1} \overset{!}{=} 0 \quad \Rightarrow \quad x = \pm 1$$

$$U''(x) = -f'(x) = \frac{-4x}{(x^2+1)^2} = \begin{cases} -1 \; < \; 0 \;\text{ für }\; x = 1 & \Rightarrow \quad \text{Max.} \quad \Rightarrow \quad \text{Sattel} \\ \;\;1 \; > \; 0 \;\text{ für }\; x = -1 & \Rightarrow \quad \text{Min.} \quad \Rightarrow \quad \text{Zentrum} \end{cases}$$

$\Rightarrow$ Zentrum in $P_1 = (-1, 0)$, Sattelpunkt in $P_2 = (1, 0)$.

Phasenkurven:

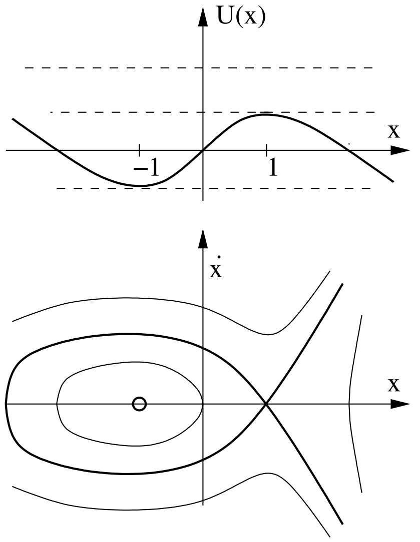
$$\int \frac{x^2-1}{x^2+1}\, dx = \ldots = x - 2\arctan x + E$$

$$\underline{\underline{\dot{x} = \pm\sqrt{2(x - 2\arctan x + E)}}}$$

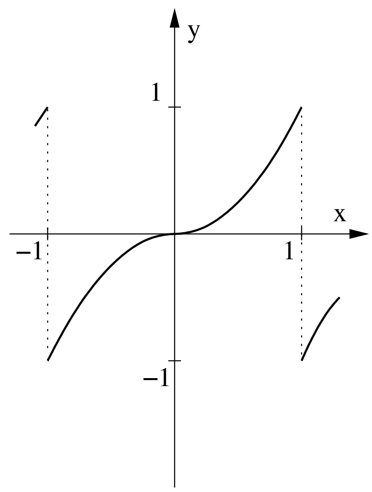
Speziell für Seperatrix (Phasenkurve durch den Sattelpunkt P_2):

$$0 = \sqrt{2(1 - 2\arctan(1) + E)} \quad \Rightarrow \quad E = -1 + \frac{\pi}{2} \doteq 0.57$$

$$\dot{x} = \sqrt{2(x - 2\arctan(x) + 0.57)}$$



Bsp. 3



Bsp. 5

zu 4.) Zeige Orthogonalität:

$$\langle f_0, f_1 \rangle = \int_{-1}^{1} 1 \cdot x\; dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0 \qquad \langle f_0, f_2 \rangle = \int_{-1}^{1} 1 \cdot (3x^2 - 1)\; dx = \left[x^3 - x\right]_{-1}^{1} = 0$$

$$\langle f_1, f_2 \rangle = \int_{-1}^{1} x \cdot (3x^2 - 1)\; dx = \left[\frac{3x^4}{4} - \frac{x^2}{2}\right]_{-1}^{1} = 0$$

Orthonormierung:

$$\langle f_0, f_0 \rangle = \int_{-1}^{1} 1 \cdot 1 dx = \left[x\right]_{-1}^{1} = 2 \quad \Rightarrow \quad \int_{-1}^{1} \frac{f_0}{\sqrt{2}} \cdot \frac{f_0}{\sqrt{2}}\; dx = 1$$

$$\langle f_1, f_1\rangle = \int_{-1}^{1} x\cdot x\; dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3} \quad\Rightarrow\quad \int_{-1}^{1} \sqrt{\frac{3}{2}}f_1 \cdot \sqrt{\frac{3}{2}}f_1\; dx = 1$$

$$\langle f_2, f_2\rangle = \int_{-1}^{1} (3x^2-1)^2\; dx = \left[\frac{9x^5}{5} - 2x^3 + x\right]_{-1}^{1} = \frac{8}{5} \quad\Rightarrow\quad \int_{-1}^{1} \sqrt{\frac{5}{8}}f_2 \cdot \sqrt{\frac{5}{8}}f_2\; dx = 1$$

Orthonormalsystem:

$$\left\{\sqrt{\frac{1}{2}}\,,\ \sqrt{\frac{3}{2}}\,x\,,\ \sqrt{\frac{5}{8}}\,(3x^2-1)\right\}$$

zu 5.) Schiefsymmetrische Funktion (Skizze s. oben) $\quad\Rightarrow\quad a_k \equiv 0.$

$$\begin{aligned}
b_k = \int_{-1}^{1} f(x)\sin k\pi x\; dx &= 2\int_0^1 \underbrace{x^2}_{u}\underbrace{\sin k\pi x\; dx}_{dv} \\
&= 2\underbrace{\left[-x^2\frac{\cos k\pi x}{k\pi}\right]_0^1}_{-(-1)^k/k\pi} + 2\int_0^1 \underbrace{2x}_{\tilde{u}}\underbrace{\frac{\cos k\pi x}{k\pi}\; dx}_{d\tilde{v}} \\
&= \frac{-2(-1)^k}{k\pi} + 2\underbrace{\left[2x\frac{\sin k\pi x}{k^2\pi^2}\right]_0^1}_{0} - 2\int_0^1 2\frac{\sin k\pi x}{k^2\pi^2}\; dx \\
&= \frac{-2(-1)^k}{k\pi} + 2\left[2\frac{\cos k\pi x}{k^3\pi^3}\right]_0^1 \\
&= \frac{-2(-1)^k}{k\pi} + 2\left[\frac{2(-1)^k}{k^3\pi^3} - \frac{2}{k^3\pi^3}\right] = \begin{cases} \frac{-2}{k\pi} & k \text{ gerade} \\ \frac{2}{k\pi} - \frac{8}{k^3\pi^3} & \text{sonst} \end{cases}
\end{aligned}$$

$$F_4(x) = \left(\frac{2}{\pi} - \frac{8}{\pi^3}\right)\sin\pi x - \frac{1}{\pi}\sin 2\pi x + \left(\frac{2}{3\pi} - \frac{8}{27\pi^3}\right)\sin 3\pi x - \frac{1}{2\pi}\sin 4\pi x$$

zu 6.) Mittels Summensatz $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$:

$$\begin{aligned}
\sin 3x &= \sin(2x+x) = \sin(2x)\cos x + \cos(2x)\sin x \\
&= 2\sin x\cos^2 x + (\cos^2 x - \sin^2 x)\sin x = 3\sin x\cos^2 x - \sin^3 x \\
&= 3\sin x(1-\sin^2 x) - \sin^3 x = 3\sin x - 4\sin^3 x
\end{aligned}$$

$$\Rightarrow\quad 3\sin x - \sin 3x = 4\sin^3 x \quad\Rightarrow\quad \sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x \checkmark$$

Das ist die Fourierreihe von $\sin^3 x$: $a_1 = 0.75$, $a_3 = -0.25$ und $a_k \equiv 0$ sonst.