**Mathematik 1 C** **WS 2005/06**

# Lösung
zur 2. Aufgabe

1. (a)

$$\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 6 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -6-4 \\ -12+3 \\ -2-12 \end{pmatrix} = \begin{pmatrix} -10 \\ -9 \\ -14 \end{pmatrix}$$

$$\left\| \begin{pmatrix} -10 \\ -9 \\ -14 \end{pmatrix} \right\| = \sqrt{100+81+196} = \underline{\underline{\sqrt{377}}} \doteq 19.4165$$

(b)

$$||\vec{a}||^2 = 1+4+4 = 9 \qquad ||\vec{b}||^2 = 36+4+9 = 49 \qquad \langle \vec{a}, \vec{b} \rangle^2 = (6-4+6)^2 = 64$$

$$\sqrt{9 \cdot 49 - 64} = \underline{\underline{\sqrt{377}}}$$

(c)

$$\langle \vec{a}, \vec{a} \rangle = 9 \qquad \langle \vec{a}, \vec{b} \rangle = 8$$

$$9 \begin{pmatrix} 6 \\ -2 \\ -3 \end{pmatrix} - 8 \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 46 \\ -34 \\ -11 \end{pmatrix} =: \vec{c}$$

$$\frac{||\vec{c}||}{||\vec{a}||} = \frac{\sqrt{46^2+34^2+11^2}}{\sqrt{9}} = \sqrt{\frac{3393}{9}} = \underline{\underline{\sqrt{377}}}$$

2.

$$c_1 \begin{pmatrix} 1 \\ 3 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ \alpha \\ 0 \end{pmatrix}$$

$$\begin{array}{rcrcrcr l}
c_1 & - & 2c_2 & & & = & 1 & \to c_2 = -\tfrac{2}{3} \\
3c_1 & & & & & = & -1 & \to c_1 = -\tfrac{1}{3} \\
 & - & 2c_2 & + & 4c_3 & = & \alpha & \to \underline{\underline{\alpha = \tfrac{8}{3}}} \\
-c_1 & + & c_2 & + & c_3 & = & 0 & \to c_3 = \tfrac{1}{3}
\end{array}$$

$$\underline{\underline{\vec{y} = -\frac{1}{3}\vec{x_1} - \frac{2}{3}\vec{x_2} + \frac{1}{3}\vec{x_3}}}$$

3. Seien $(x_1, y_1)$, $(x_2, y_2) \in \mathbb{R}^2$, $\alpha \in \mathbb{R}$.

$$\begin{aligned}
\underline{\underline{F((x_1,y_1)) + F((x_2,y_2))}} &= \begin{pmatrix} 2x_1 & x_1+y_1 \\ y_1-x_1 & 2y_1 \end{pmatrix} + \begin{pmatrix} 2x_2 & x_2+y_2 \\ y_2-x_2 & 2y_2 \end{pmatrix} \\
&= \begin{pmatrix} 2x_1+2x_2 & x_1+y_1+x_2+y_2 \\ y_1-x_1+y_2-x_2 & 2y_1+2y_2 \end{pmatrix} \\
&= \begin{pmatrix} 2(x_1+x_2) & (x_1+x_2)+(y_1+y_2) \\ (y_1+y_2)-(x_1+x_2) & 2(y_1+y_2) \end{pmatrix} \\
&= F((x_1+x_2, y_1+y_2)) \;=\; \underline{\underline{F((x_1,y_1)+(x_2,y_2))}}
\end{aligned}$$

$$\begin{aligned}
\underline{\underline{\alpha F((x_1,y_1))}} &= \alpha \begin{pmatrix} 2x_1 & x_1+y_1 \\ y_1-x_1 & 2y_1 \end{pmatrix} \\
&= \begin{pmatrix} 2\alpha x_1 & \alpha x_1+\alpha y_1 \\ \alpha y_1-\alpha x_1 & 2\alpha y_1 \end{pmatrix} \\
&= F((\alpha x_1, \alpha y_1)) \;=\; \underline{\underline{F(\alpha(x_1,y_1))}}
\end{aligned}$$

<u>Die Funktion $F$ ist linear</u>.

4.

$$\begin{pmatrix} 1 & -4 & 2 \\ 3 & 1 & 0 \\ 1 & 9 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -4 & 2 \\ 0 & 13 & -6 \\ 0 & 13 & -6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -4 & 2 \\ 0 & 13 & -6 \\ 1 & 0 & 0 \end{pmatrix} \rightarrow \underline{\underline{\mathrm{Rg}(A) = 2}}$$

$$\begin{pmatrix} 2 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & -1 & -13/2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & -15/2 \end{pmatrix} \rightarrow \underline{\underline{\mathrm{Rg}(B) = 3}}$$

$$A \cdot B = \begin{pmatrix} 1 & -4 & 2 \\ 3 & 1 & 0 \\ 1 & 9 & -4 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -6 & -1 \\ 8 & 9 & 11 \\ 8 & 21 & 13 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 8 & 9 & 11 \\ 0 & -6 & -1 \\ 8 & 21 & 13 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 8 & 9 & 11 \\ 0 & -6 & -1 \\ 0 & 12 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 8 & 9 & 11 \\ 0 & -6 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \underline{\underline{\mathrm{Rg}(A \cdot B) = 2}}$$

5.

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & \alpha & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & \alpha & 0 & 0 & 1 \\ 2 & 1 & -1 & 0 & 1 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & \alpha & 0 & 0 & 1 \\ 0 & 5 & -1 & -2 & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & \alpha & 0 & 0 & 1 \\ 0 & 0 & -1-5\alpha & -2 & 1 & -5 \end{array}\right)$$

Die Matrix ist *regulär* und damit *invertierbar*, wenn $-1 - 5\alpha \neq 0$, also $\underline{\underline{\alpha \neq -\frac{1}{5}}}$. Dann können wir die letzte Zeile durch $-1 - 5\alpha$ dividieren und weiterrechnen:

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & \alpha & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{2}{1+5\alpha} & \frac{-1}{1+5\alpha} & \frac{5}{1+5\alpha} \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{-2\alpha}{1+5\alpha} & \frac{\alpha}{1+5\alpha} & \frac{1}{1+5\alpha} \\ 0 & 0 & 1 & \frac{2}{1+5\alpha} & \frac{-1}{1+5\alpha} & \frac{5}{1+5\alpha} \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1+\alpha}{1+5\alpha} & \frac{2\alpha}{1+5\alpha} & \frac{2}{1+5\alpha} \\ 0 & 1 & 0 & \frac{-2\alpha}{1+5\alpha} & \frac{\alpha}{1+5\alpha} & \frac{1}{1+5\alpha} \\ 0 & 0 & 1 & \frac{2}{1+5\alpha} & \frac{-1}{1+5\alpha} & \frac{5}{1+5\alpha} \end{array}\right) \rightarrow \underline{\underline{A^{-1} = \frac{1}{1+5\alpha} \begin{pmatrix} 1+\alpha & 2\alpha & 2 \\ -2\alpha & \alpha & 1 \\ 2 & -1 & 5 \end{pmatrix}}}$$