Variation der Konstanten

B Man löse die Differentialgleichung

$$y'' - 8y' + 16y = \underbrace{e^{4x}\sqrt{x}}_{f(x)} \qquad y(1) = 1, \quad y'(1) = -2$$

1. Homogene Lösung:

$$\lambda^2 - 8\lambda + 16 = 0 \quad \Rightarrow \quad \lambda_{1,2} = 4 \text{ (Resonanz!)}$$

$$\underline{\underline{y_h(x) = c_1 e^{4x} + c_2\, x\, e^{4x}}}$$

2. Partikulärlösung, Variation der Konstanten:

$$y_p(x) = c_1(x)\underbrace{e^{4x}}_{y_1(x)} + c_2(x)\underbrace{xe^{4x}}_{y_2(x)}$$

wobei

$$c_1(x) = \int \frac{-y_2(x) f(x)\, dx}{W(x)} \qquad\qquad c_2(x) = \int \frac{y_1(x) f(x)\, dx}{W(x)}$$

$$\text{mit } W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^{4x} & xe^{4x} \\ 4e^{4x} & (1-4x)e^{4x} \end{vmatrix} = (1-4x)e^{8x} - 4xe^{8x} = e^{8x}$$

$$c_1(x) = \int \frac{-xe^{4x}e^{4x}\sqrt{x}\, dx}{e^{8x}} = -\int x^{3/2}\, dx = -\frac{2x^{5/2}}{5}$$

$$c_2(x) = \int \frac{e^{4x}e^{4x}\sqrt{x}\, dx}{e^{8x}} = \int \sqrt{x}\, dx = \frac{2x^{3/2}}{3}$$

$$y_p(x) = -\frac{2x^{5/2}}{5}e^{4x} + \frac{2x^{3/2}}{3}xe^{4x} = \underline{\underline{\frac{4}{15}x^{5/2}e^{4x}}}$$

3. Allgemeine Lösung:

$$y(x) = c_1 e^{4x} + c_2 x e^{4x} + \frac{4}{15}x^{5/2}e^{4x} = \underline{\underline{\left(c_1 + c_2 x + \frac{4}{15}x^{5/2}\right)e^{4x}}}$$

4. Lösung des Anfangswertproblems:

$$y(1) = \left(c_1 + c_2 + \frac{4}{15}\right)e^4 = 1$$

$$y'(x) = \left(c_2 + \frac{4}{15}\cdot\frac{5}{2}x^{3/2}\right)e^{4x} + \left(c_1 + c_2 x + \frac{4}{15}x^{5/2}\right)\cdot 4e^{4x} = \left(4c_1 + c_2 + 4c_2 x + \frac{2}{3}x^{3/2} + \frac{16}{15}x^{5/2}\right)e^{4x}$$

$$y'(1) = \left(4c_1 + 5c_2 + \frac{2}{3} + \frac{16}{15}\right)e^4 = -2$$

$$c_1 + c_2 = e^{-4} - \frac{4}{15} \doteq -0.248$$

$$4c_1 + 5c_2 = -2e^{-4} - \frac{2}{3} - \frac{16}{15} \doteq -1.770$$

$$\Rightarrow \quad c_1 \doteq 0.530 \qquad c_2 \doteq -0.778$$

$$\underline{\underline{y(x) = \left(0.530 - 0.778x + \frac{4}{15}x^{5/2}\right)e^{4x}}}$$