$\boxed{\text{B}}$ Man bestimme die Lösung des folgenden Dgl-Systems

$$\begin{array}{rcrcrcl} x' & = & -4x & - & 2y & + & e^{2t} \\ y' & = & 9x & + & 2y & + & 1 \end{array} \qquad \begin{array}{l} x(0)=1 \\ y(0)=0 \end{array}$$

durch Anwendung der $\mathcal{L}$-Transformation.

1. Die $\mathcal{L}$-Transformation der einzelnen Komponenten liefert:

$$\mathcal{L}\{x(t)\} = X(s) \qquad \mathcal{L}\{y(t)\} = Y(s) \qquad \mathcal{L}\{e^{2t}\} = \tfrac{1}{s-2}$$

$$\mathcal{L}\{x'(t)\} = sX(s) - 1 \qquad \mathcal{L}\{y'(t)\} = sY(s) - 0 \qquad \mathcal{L}\{1\} = \tfrac{1}{s}$$

2. Transformation des Gleichungssystems:

$$\begin{aligned} sX(s) - 1 &= -4X(s) - 2Y(s) + \frac{1}{s-2} \\ sY(s) &= 9X(s) + 2Y(s) + \frac{1}{s} \end{aligned}$$

3. Umformung des Gleichungssystems:

$$\begin{aligned} (s+4)X(s) + 2\,Y(s) &= \frac{s-1}{s-2} \\ -9\,X(s) + (s-2)Y(s) &= \frac{1}{s} \end{aligned}$$

4. Auflösen nach $X(x)$ bzw. $Y(s)$:

   (a) Durch Addition des 9-fachen der ersten zum $(s+4)$-fachen der zweiten Zeile wird $X(s)$ eliminiert:

$$\begin{aligned} 18Y(s) + (s+4)(s-2)Y(s) &= \frac{9(s-1)}{s-2} + \frac{(s+4)}{s} \\ (s^2+2s+10)Y(s) &= \frac{10s^2-7s-8}{s(s-2)} \\ Y(s) &= \frac{10s^2-7s-8}{s(s-2)(s^2+2s+10)} \end{aligned}$$

   (b) Einsetzen in die erste Zeile von Punkt 3. liefert $X(s)$:

$$\begin{aligned} (s+4)X(s) + 2\,\frac{10s^2-7s-8}{s(s-2)(s^2+2s+10)} &= \frac{s-1}{s-2} \\ (s+4)X(s) &= \frac{s^4+s^3-12s^2+4s+16}{s(s-2)(s^2+2s+10)} \\ X(s) &= \frac{s^4+s^3-12s^2+4s+16}{s(s-2)(s+4)(s^2+2s+10)} \end{aligned}$$

5. Partialbruchzerlegungen:

$$
\begin{aligned}
X(s) \;=\; \frac{s^4+s^3-12s^2+4s+16}{s(s-2)(s+4)(s^2+2s+10)} &= \frac{A_1}{s}+\frac{B_1}{s-2}+\frac{C_1}{s+4}+\frac{D_1+sE_1}{s^2+2s+10} \\
&= -\frac{1}{5}\cdot\frac{1}{s}-\frac{3}{5}\cdot\frac{1-2s}{s^2+2s+10} \\
Y(s) \;=\; \frac{10s^2-7s-8}{s(s-2)(s^2+2s+10)} &= \frac{A_2}{s}+\frac{B_2}{s-2}+\frac{C_2+sD_2}{s^2+2s+10} \\
&= \frac{2}{5}\cdot\frac{1}{s}+\frac{1}{2}\cdot\frac{1}{s-2}+\frac{9}{10}\cdot\frac{8-s}{s^2+2s+10}
\end{aligned}
$$

6. benötigte inverse $\mathcal{L}$-Transformationen:

$$
\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \qquad\qquad \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}
$$

$$
\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+10}\right\} = \frac{1}{\mathbf{3}}\mathcal{L}^{-1}\left\{\frac{1\cdot\mathbf{3}}{(s+1)^2+9}\right\} = \frac{1}{3}e^{-t}\sin 3t
$$

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{s\mathbf{+1}-\mathbf{1}}{(s+1)^2+9}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+9}\right\} \\
&= e^{-t}\cos 3t-\frac{1}{3}e^{-t}\sin 3t
\end{aligned}
$$

7. Rücktransformationen:

$$
\begin{aligned}
\underline{\underline{x(t)}} &= \mathcal{L}^{-1}\left\{-\frac{1}{5}\cdot\frac{1}{s}-\frac{3}{5}\cdot\frac{1}{s^2+2s+10}+\frac{6}{5}\cdot\frac{s}{s^2+2s+10}\right\} \\
&= -\frac{1}{5}-\frac{\cancel{3}}{5}\cdot\frac{1}{\cancel{3}}e^{-t}\sin 3t+\frac{6}{5}\left(e^{-t}\cos 3t-\frac{1}{3}e^{-t}\sin 3t\right) \\
&= \underline{\underline{-\frac{1}{5}+\frac{6}{5}e^{-t}\cos 3t-\frac{3}{5}e^{-t}\sin 3t}} \\
\underline{\underline{y(t)}} &= \mathcal{L}^{-1}\left\{\frac{2}{5}\cdot\frac{1}{s}+\frac{1}{2}\cdot\frac{1}{s-2}+\frac{36}{5}\cdot\frac{1}{s^2+2s+10}-\frac{9}{10}\cdot\frac{s}{s^2+2s+10}\right\} \\
&= \frac{2}{5}+\frac{1}{2}e^{2t}+\frac{\cancel{36}^{12}}{5}\cdot\frac{1}{\cancel{3}}e^{-t}\sin 3t-\frac{9}{10}\left(e^{-t}\cos 3t-\frac{1}{3}e^{-t}\sin 3t\right) \\
&= \underline{\underline{\frac{2}{5}+\frac{1}{2}e^{2t}-\frac{9}{10}e^{-t}\cos 3t+\frac{27}{10}e^{-t}\sin 3t}}
\end{aligned}
$$