On the structure of groups supporting a number system

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Abstract

Let G be an abelian group, f an endomorphism of G and D a subset of G such that $\#D = [G : f(G)]$. I call (G, f, D) a number system if every element of G has a finite expansion of the form

$$g = \sum_{i=0}^{\ell} f^i(d_i) \qquad (d_i \in D).$$

We may ask which groups G support such a number system. I will give some examples, and prove some finiteness properties that G must have. Next, I will propose the conjecture that G must be a split group, and comment on the possibility to prove this claim.

Part of this work is joint with Ryotaro Okazaki.

Introduction

Considering, for example,

- canonical number systems,
- digital expansions in $\mathbb{F}_q[x]$,
- β-expansions,

we see that many numeration systems share the underlying algebraic structure of an abelian group with an endomorphism whose image has finite index.

We will explore this common structure in more detail, to see if this leads to essentially new numeration systems.

Definitions

Let G be an abelian group, and $f : G \to G$ an endomorphism.

Suppose $[G : f(G)]$ is finite, and let $D \subseteq G$ contains a system of representatives for G modulo $f(G)$; so $\#D \geq [G : f(G)]$.

We call (G, f, D) a number system if every $g \in G$ has a finite expansion of the form

$$g = \sum_{i=0}^{\ell} f^i(d_i) \quad (d_i \in D).$$

Recall that $g \in G$ is a torsion element if there exists an integer $n \neq 0$ such that $ng = 0$. If no such n exists, then g is torsion-free. An abelian group G is torsion-free (a torsion group) if all its elements are torsion-free (torsion elements).

Examples

A canonical number system, say $(\mathbb{Z}[\alpha], \alpha, \{0, 1, \ldots, |\operatorname{Norm}(\alpha)| - 1\})$ for a suitable algebraic integer α, is an example; here the group $\mathbb{Z}[\alpha]$ is a finitely generated free abelian group.

Recall that a finitely generated torsion-free abelian group is automatically free, and is commonly called a (finite-dimensional) lattice.

The "function field analogon" of a canonical number system is $(\mathbb{F}[x][y]/(P), y, \{f \in \mathbb{F}[x] : \deg(f) < \deg(P(0))\})$, where $\mathbb{F}$ is a finite field and $P \in \mathbb{F}[x][y]$ is monic. Here the group $\mathbb{F}[x][y]/(P)$ is an infinite-dimensional vector space over $\mathbb{F}$; it is a torsion group of finite exponent $p = \operatorname{char} \mathbb{F}$, generated by

$$\{x^i y^j : i \geq 0, 0 \leq j < \deg(P)\}.$$

Main question

A mixed group is an abelian group that is neither torsion-free nor a torsion group. A trivial example is

$$\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}).$$

Let G be an abelian group. The set of all torsion elements of G is a subgroup G^{tor}. If we have

$$G \cong G^{\text{tor}} \oplus H$$

for some subgroup $H \subseteq G$, then G is split. In this case, H is isomorphic to the torsion-free quotient G/G^{tor}.

By the fundamental theorem, all finitely generated abelian groups are split.

Main question: if G supports a number system (G, f, D), must G be split?

Direct products

If (G, f, D) and (G', f', D') are number systems, then clearly so is

$$(G \times G', f \times f', D \times D').$$

Of course, the underlying group here is split.

There may be number systems in $G \times G'$ where the endomorphism or the digit set is not split, but first we ask if the group itself is split.

If (G, f, D) is a number system and H is an f-invariant subgroup, then

$$(G/H, f, D/H)$$

is also a number system; however, the digit set D become redundant modulo H.

Now G^{tor} is certainly f-invariant for any f, so there is an induced number system on G/G^{tor}.

Some finiteness results

Let G be an abelian group supporting a number system (G, f, D).

Lemma 1: G is countable.

Lemma 2: G is generated by $\{f^i(d) : i \geq 0, d \in D\}$.

Lemma 3: if G is torsion, then G is bounded.

The torsion-free quotient G/G^{tor} is contained in the $\mathbb{Q}$-vector space $V \cong \mathbb{Q} \otimes G$, and the dimension of V is called the (torsion-free) rank of G.

Theorem (Okazaki-vdW) G has finite torsion-free rank and the kernel of f is contained in G^{tor}.

This means that G/G^{tor} is contained in a finite-dimensional $\mathbb{Q}$-vector space V, and on it, f is given by a finite square matrix with rational entries.

Proof

Assume G is torsion-free.

Let $\Delta = [G : f(G)]$; then for every digit d, $\Delta d = f(d')$ for some d'. Expand these d' in our number system; maximal length L. Let

$$W = \langle f^i(d) : d \in D, 0 \le i \le L - 1 \rangle \subseteq G;$$

then $\Delta W \subseteq f(W)$. Thus, f is an isomorphism on $W \otimes \mathbb{Q}$.

It follows that $f^i(d) \in W \otimes \mathbb{Q}$ for all d and all i, so

$$G \subseteq W \otimes \mathbb{Q}.$$

But the rank of W is at most $\#D \cdot L$.

Also, if $v \in \ker f$, then the image of f has smaller rank than G, contradiction.

Abelian group theory

Some fascinating theorems:

Theorem (Baer, Fomin) For a torsion group T, the property that every abelian group G with $G^{\text{tor}} \cong T$ is split is equivalent to T being a direct sum of a divisible group and a bounded group.

Theorem (Baer, Griffith) For a torsion-free group H, the property that every abelian group G with $G/G^{\text{tor}} \cong H$ is split is equivalent to H being free.

Unfortunately, if (G, f, D) is a number system, G/G^{tor} need not be free.

Example: $G = \mathbb{Z}[\frac{1}{2}]$, with the endomorphism multiplication by 5/2 and digits $D = \{-2, -1, 0, 1, 2\}$. Here G is itself torsion-free (it is contained in $\mathbb{Q}$), but not free. In fact, G is 2-divisible.

Also, we do not know that G^{tor} must be bounded $\oplus$ divisible.

(No) Pathological cases

Let H be a countable torsion-free group and T a torsion group. There is a very technical result by Baer that gives a necessary and sufficient condition for $\mathrm{Ext}(H,T)$ to be trivial. (This means that every group G with $G^{\mathrm{tor}} \cong T$ and $G/G^{\mathrm{tor}} \cong H$ is split.)

The first criterion:

> *if $p_1, \ldots, p_n, \ldots$ is an infinite set of different primes for which $p_i T$ is strictly contained in T, then H contains no pure subgroup S of finite rank such that H/S has elements $\neq 0$ divisible by all p_i.*

This situation can be excluded, because our endomorphism f has a finite denominator, and all bad primes must divide it.

(No) pathological cases (2)

The second criterion:

> *if for some prime p, the reduced part of the p-component of T is unbounded, then H contains no pure subgroup S of finite rank such that H/S has elements $\neq 0$ divisible by all powers of p.*

Difficult problem: can we exclude this situation?

If we assume that G has torsion-free rank 1, then if G is nonsplit, the quotient G/G^{tor} must be p-divisible for some bad prime p.

But recall the 2-divisible example I gave earlier.

Some good news

An example from Fuchs' book: let

$$T_p = \oplus_{i=1}^{\infty} \langle a_i \rangle, \text{ with } \mid \langle a_i \rangle \mid = p^{2i};$$

$$b_i = (0, \ldots, 0, a_i, pa_{i+1}, p^2 a_{i+2}, \ldots) \in \prod_{i=1}^{\infty} \langle a_i \rangle;$$

now let

$$A_p = \langle T_p, b_1, b_2, \ldots \rangle.$$

Then the torsion part T_p is not a direct summand of A_p. Note that

$$A_p / T_p \cong \mathbb{Z}[1/p]^+.$$

Theorem The groups A_p do not support a number system.

Proof of the example.

The b_i, as defined above, are of infinite order and satisfy $pb_{i+1} = b_i - a_i$ for $i = 1, 2, \ldots$. Using these relations, it is readily checked that T_p is the torsion part of A_p.

If we had $A_p = T_p \oplus G$ for some subgroup G of A_p, then because $pb_{i+1} \equiv b_i \mod T_p$, the group G would be p-divisible, contrary to the fact that $\prod \langle a_i \rangle$ has no p-divisible subgroups $\neq 0$.

Something on the proof

The main idea of the proof is showing that, if f is an endomorphism of A_p, then its induced endomorphism on A_p/T_p has no factor p in its denominator. This means that the successive images of some finitely generated subgroup of A_p/T_p, like the subgroup generated by some finite digit set D, cannot fill up the whole group.

If we can make this proof work generally, then we may approach the following (however, a classification of mixed groups is only known for the case of rank 1).

Conjecture Any abelian group supporting a number system is split.

Also, if we replace $\mathbb{Z}$ by a more general ground ring $\mathcal{E}$ of dimension 1, we have the analogous conjecture with abelian groups replaced by $\mathcal{E}$-modules.