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THOMAS' FAMILY OF THUE EQUATIONS OVER IMAGINARY QUADRATIC FIELDS

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Dedicated to Professor H. G. Zimmer on his 65th birthday

Abstract. We consider the family of relative Thue equations

$$x^3 - (t-1)x^2y - (t+2)xy^2 - y^3 = \mu,$$

where the parameter t, the root of unity μ and the solutions x and y are integers in the same imaginary quadratic number field.

We prove that there are only trivial solutions (with $|x|$, $|y| \leq 1$), if $|t|$ is large enough or if the discriminant of the quadratic number field is large enough or if $\operatorname{Re} t = -1/2$ (there are a few more solutions in this case which are explicitly listed). In the case $\operatorname{Re} t = -1/2$, an algebraic method is used, in the general case, Baker's method yields the result.

1. Introduction

Let $F \in \mathbb{Z}[x, y]$ be an irreducible form of degree at least 3 and m be a nonzero integer. Then the Diophantine equation

$$F(x, y) = m$$

is called a *Thue* equation in honor of A. Thue [21] who proved that it has only finitely many solutions x, $y \in \mathbb{Z}$. Upper bounds for the solutions have been given using A. Baker's [1] theory on linear forms in logarithms of algebraic numbers, cf. Bugeaud and Győry [4]. Algorithms to solve single Thue equations have been developed, we refer to Bilu and Hanrot [3].

In the last decade, several families of parametrized Thue equations have been investigated; a survey containing further references is given in [10]. Furthermore, we refer to the more recent results [23], [20], [22], [6], [12], [5], [8], and [11].

Relative Thue equations, i. e. Thue equations with coefficients in an algebraic number field, where solutions come from the ring of integers of the same number field, have also been successfully solved, we refer to Gaál and Pohst [9] for further references.

However, to our knowledge, no infinite family of relative Thue equations has been solved up to now. The aim of this paper is to solve such a family. As the ground field, we take an imaginary quadratic number field. We consider the form $x^3 - (t-1)x^2y - (t+2)xy^2 - y^3$. If $t \in \mathbb{Z}$, it generates Shanks' [17] simplest cubic number fields. The corresponding family of absolute Thue equations was the first family of positive discriminant which has been solved, cf. Thomas [18] and Mignotte [14]. We consider the same form with an imaginary quadratic integral parameter t. This setting has been chosen in order to avoid the unit rank to increase.

Our main result is the following:

Theorem 1. *Let t be an integer in an imaginary quadratic number field, $\mathbb{Z}_{\mathbb{Q}(t)}$ be the ring of integers of $\mathbb{Q}(t)$ and μ be a root of unity in $\mathbb{Z}_{\mathbb{Q}(t)}$.*

If $|t| \geq 3\,023\,195\,238$, then the only solutions x, y in $\mathbb{Z}_{\mathbb{Q}(t)}$ to the family of relative Thue equations

$$x^3 - (t-1)x^2y - (t+2)xy^2 - y^3 = \mu \tag{1}$$

are listed in Table 1.

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x	y	μ	x	y	μ	x	y	μ
0	1	-1	i	0	$-\mathrm{i}$	$-1+\omega_3$	$1-\omega_3$	-1
-1	0	-1	0	i	i	ω_3	0	-1
1	-1	-1	$-\mathrm{i}$	0	i	0	$1-\omega_3$	1
0	-1	1	i	$-\mathrm{i}$	i	0	ω_3	1
-1	1	1	0	$-\omega_3$	-1	$-\omega_3$	0	1
1	0	1	0	$-1+\omega_3$	-1	$1-\omega_3$	$-1+\omega_3$	1
0	$-\mathrm{i}$	$-\mathrm{i}$	$-\omega_3$	ω_3	-1	$-1+\omega_3$	0	1
$-\mathrm{i}$	i	$-\mathrm{i}$	$1-\omega_3$	0	-1	ω_3	$-\omega_3$	1

TABLE 1. Solutions (if contained in $\mathbb{Q}(t)$) to (1) for all t, where $\omega_3 = (1+\sqrt{-3})/2$.

For $t \in \mathbb{Z}^{\geq 0}$, Theorem 1 has been proved by Thomas [18]. Writing $F_t(X,Y) = X^3-(t-1)X^2Y-(t+2)XY^2-Y^3$ and noting that $F_{-t-1}(Y,X) = -F_t(X,Y)$, his result is easily generalized to negative rational integers t. We note that Thomas [18] and Mignotte [14] solved the rational integer case completely; they found some extra solutions for $|t| \leq 4$. Therefore, we will assume $t \notin \mathbb{Z}$ in the remainder of this paper.

The condition $|t| \geq 3\,023\,195\,238$ is certainly fulfilled if the discriminant of the base field is sufficiently large (in absolute value). It is well known that if $D_K < -4$ then the only roots of unity in K are ± 1. Hence, if $|D_K| > 4$ we may take $\mu = \pm 1$ and get immediately the following corollary.

Corollary 2. *Let k be an imaginary quadratic number field with discriminant $\leq -3.66 \cdot 10^{19}$ and $t \in k \setminus \mathbb{Z}$ be an algebraic integer. Then the only solutions x, $y \in \mathbb{Z}_k$ to*

$$x^3-(t-1)x^2y-(t+2)xy^2-y^3 = \pm 1$$

are $\pm\{(0,1),(-1,0),(1,-1)\}$.

The proof of Theorem 1 is based on Baker's theory on linear forms in logarithms of algebraic numbers. This yields the rather large constant $3.03 \cdot 10^9$. Since the parameter t is complex, a complete enumeration of the remaining cases seems to be hopeless. However, if $\operatorname{Re} t = -1/2$, we can use an algebraic argument and solve the equation completely for these values of the parameter.

Theorem 3. *Let $t \neq (-1 \pm 3\sqrt{-3})/2$ be an integer in an imaginary quadratic number field with $\operatorname{Re} t = -1/2$ and μ be a root of unity in $\mathbb{Q}(t)$. Then the only solutions x, $y \in \mathbb{Z}_{\mathbb{Q}(t)}$ to (1) are listed in Table 1 or in Table 2.*

t	$(x,y) \in$
$-\omega_3$	$\pm\{(3-6\omega_3,-1+3\omega_3),(-2+3\omega_3,3-6\omega_3),(-1+3\omega_3,-2+3\omega_3)\}$
$-1+\omega_3$	$\pm\{(-3+6\omega_3,2-3\omega_3),(1-3\omega_3,-3+6\omega_3),(2-3\omega_3,1-3\omega_3)\}$
$-\omega_7$	$\pm\{(-1+2\omega_7,-\omega_7),(-\omega_7,1-\omega_7),(1-\omega_7,-1+2\omega_7)\}$
$-1+\omega_7$	$\pm\{(1-2\omega_7,-1+\omega_7),(-1+\omega_7,\omega_7),(\omega_7,1-2\omega_7)\}$
$-\omega_{19}$	$\pm\{(-1+2\omega_{19},-1-\omega_{19}),(-1-\omega_{19},2-\omega_{19}),(2-\omega_{19},-1+2\omega_{19})\}$
$-1+\omega_{19}$	$\pm\{(1-2\omega_{19},-2+\omega_{19}),(-2+\omega_{19},1+\omega_{19}),(1+\omega_{19},1-2\omega_{19})\}$

TABLE 2. All solutions for $\operatorname{Re} t = -1/2$ to (1), where $\omega_D = (1+\sqrt{-D})/2$.

The case $t = (-1 \pm 3\sqrt{-3})/2$ has to be excluded because the form F_t is the cube of a linear polynomial in that case.

The remainder of this paper is organized as follows: In Section 2, we collect elementary properties and asymptotic expressions for the algebraic numbers generating the splitting field of F_t. Small solutions are dealt with in Section 3. The Galois structure of the family will be determined in Section 4. It will turn out that the case $\operatorname{Re} t = -1/2$ has a special Galois group. The unit structure of the relevant order will be discussed in Section 5. Approximation properties of solutions are the topic of Section 6. "Stable growth" will be proved in Section 7, which will exclude all medium sized solutions. The large solutions are excluded in Section 8 using a corollary of Mignotte [15] of the lower bound for linear forms in two logarithms by Laurent, Mignotte, and Nesterenko [13]. Finally, we will prove Theorem 3 in Section 9.

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2. Elementary Properties

Let $t \in \mathbb{C}$ and

$$F_t(X,Y) := X^3 - (t-1)X^2Y - (t+2)XY^2 - Y^3, \tag{2}$$

$$f_t(X) := F_t(X,1). \tag{3}$$

It is straightforward to verify the identities

$$F_t\big(Y, -(X+Y)\big) = F_t\big(-(X+Y), X\big) = F_t(X,Y) \qquad \text{and} \qquad f_t\left(-1-\frac{1}{X}\right) = \frac{f_t(X)}{X^3}. \tag{4}$$

Let $\alpha^{(1)} = \alpha = \alpha(t)$ be a root of $f_t(X) = 0$. Then (4) implies that

$$\alpha^{(2)} := -1 - \frac{1}{\alpha} \qquad \text{and} \qquad \alpha^{(3)} := -1 - \frac{1}{\alpha^{(2)}} = -\frac{1}{\alpha+1} \tag{5}$$

are roots of f_t. The roots α, $\alpha^{(2)}$, and $\alpha^{(3)}$ are pairwise distinct for $t \neq (-1 \pm 3\sqrt{-3})/2$. Indeed, assuming $\alpha = \alpha^{(2)}$ implies $\alpha^2 + \alpha + 1 = f_t(\alpha) = 0$, which is possible only if $t = (-1 \pm 3\sqrt{-3})/2$. If $t = (-1 \pm 3\sqrt{-3})/2$, we have $f_t(X) = (X - (t-1)/2)^3$.

Assume that $\alpha \in \mathbb{R}$. Then we get $0 = f_t(\alpha) - \overline{f_t}(\alpha) = -(t - \overline{t})(\alpha^2 + \alpha)$, which implies $t \in \mathbb{R}$.

We will use the following variant of the usual O-notation: For two functions $g(t)$ and $h(|t|)$ and a positive number t_0 we will write $g(t) = L_{t_0}\big(h(|t|)\big)$ if $|g(t)| \leq h(|t|)$ for all t with absolute value at least t_0. We will use this notation in the middle of an expression in the same way as it is usually done with the O-notation.

Lemma 4. *Let $t \in \mathbb{C}$. Then there is a root α of f_t such that*

$$\alpha = t + \frac{2}{t} - \frac{1}{t^2} - \frac{3}{t^3} + L_6\left(\frac{5}{|t|^{7/2}}\right). \tag{6}$$

Proof. Let $h(z) = f(z + t + 2t^{-1} - t^{-2} - 3t^{-3})$ and $h_1(z) := h(z) - h(0)$. For $|t| \geq 6$ and $|z| = 5\,|t|^{-7/2}$, we obtain

$$|h(z) - h_1(z)| = |h(0)| < \frac{7.38}{|t|^2} < \frac{5}{|t|^{3/2}} - \frac{10.26}{|t|^{5/2}} < |h_1(z)|\,.$$

By Rouché's theorem, h and h_1 have the same number of zeros in the disc $\{z : |z| < 5\,|t|^{-7/2}\}$. Since $h_1(0) = 0$, this implies the theorem. □

Similarly, we will use

$$\alpha = t + \frac{2}{t} - \frac{1}{t^2} - \frac{3}{t^3} + \frac{5}{t^4} + \frac{7}{t^5} - \frac{26}{t^6} - \frac{10}{t^7} + L_{20}\left(\frac{33}{|t|^{15/2}}\right). \tag{7}$$

Denoting the root calculated in Lemma 4 by $\alpha^{(1)} = \alpha$ and using (5), we get the following estimates for the other roots:

$$\alpha^{(2)} = -1 - \frac{1}{\alpha^{(1)}} = -1 - \frac{1}{t} + L_6\left(\frac{102}{5\,|t|^{5/2}}\right), \tag{8a}$$

$$\alpha^{(3)} = -\frac{1}{\alpha^{(1)}+1} = -\frac{1}{t} + \frac{1}{t^2} + L_6\left(\frac{183}{10\,|t|^{5/2}}\right). \tag{8b}$$

3. Small Solutions

Lemma 5. *Let $|t| > 4$. All solutions $(x, y) \in \mathbb{Z}_{\mathbb{Q}(t)}$ to (1) with $|y| < \sqrt{5}$, $|x + y| < \sqrt{5}$, or $|x| < \sqrt{5}$ are listed in Table 1.*

Proof. We will first consider solutions with $|y| < \sqrt{5}$.

Let some imaginary quadratic root of unity μ and some imaginary quadratic integer y with $|y| < \sqrt{5}$ be given such that $[\mathbb{Q}(\mu, y) : \mathbb{Q}] \leq 2$. It is clear that there is only a finite number of such pairs (y, μ). We are looking for quadratic integers x and t such that $[\mathbb{Q}(t, x, y, \mu) : \mathbb{Q}] \leq 2$ and $F_t(x, y) = \mu$. The latter condition is equivalent to

$$x(x^2 - (t-1)xy - (t+2)y^2) = \mu + y^3. \tag{9}$$

If $\mu + y^3 \neq 0$, we conclude that $N(x) \mid N(\mu + y^3)$, where $N(\,\cdot\,) = N_{\mathbb{Q}(t)/\mathbb{Q}}(\,\cdot\,)$ denotes the norm function. This yields a finite number of possible values for x. We only consider those x such that $[\mathbb{Q}(\mu, x, y) : \mathbb{Q}] \leq 2$ and such that $(\mu + y^3)/x$ is an algebraic integer.

Then (9) implies $t(-xy - y^2) = (\mu + y^3)/x - x^2 - xy + 2y^2$. If $(-xy - y^2) = (\mu + y^3)/x - x^2 - xy + 2y^2 = 0$, we found a solution for all values of t and list it in Table 1. Otherwise, if $xy + y^2 \neq 0$, we calculate an explicit value for t. We check whether this t is an algebraic integer. It turns out that this process only yields values of t with $|t| \leq 4$.

If $\mu + y^3 = 0$, we may either have $x = 0$ and arbitrary t (in the appropriate field) and the solution is listed in Table 1, or (9) results in

$$(x + y)(x - yt) = 2y^2.$$

This implies $N(x + y) \mid N(2y^2) = 4$. Therefore, we only have a finite number of choices for $x + y$. For each choice, we get a value of t. It turns out that all those t have absolute value at most 4.

Let now (x, y) be a solution of (1) with $|x| < \sqrt{5}$. By (4), $\bigl(-(x + y), x\bigr)$ is a solution contained in Table 1. The case $|x + y| < \sqrt{5}$ is dealt with similarly. It can be checked that for each solution (x, y) listed in Table 1, $\bigl(y, -(x + y)\bigr)$ and $\bigl(-(x + y), x\bigr)$ are also contained in the table. □

4. Computation of the Galois Group

For the remainder of this paper, we fix a positive squarefree integer D and assume that $t \in \mathbb{Z}_k \backslash \mathbb{Z}$, where $\mathbb{Z}_k$ is the ring of integers in the algebraic number field $k := \mathbb{Q}(\sqrt{-D})$.

Lemma 6. *For $t \neq (-1 \pm 3\sqrt{-3})/2$, f_t is irreducible over k.*

Proof. Assume first $|t| \geq 6$. From (8) and $\alpha^{(3)} \notin \mathbb{R}$ we see that $0 < |\alpha^{(3)}| \leq 1/2$ which implies that $\alpha^{(3)} \notin \mathbb{Z}_k$. However, $\alpha^{(3)}$ is an algebraic integer. This yields $\alpha^{(3)} \notin k$. From (5) we conclude that $\alpha^{(j)} \notin k$ for $j = 1, \ldots, 3$.

For $|t| < 6$, the assertion has been checked using Pari [2]. □

Let $K := \mathbb{Q}(\alpha)$. From (5) and $t - 1 = \alpha^{(1)} + \alpha^{(2)} + \alpha^{(3)}$ we see that $K = \mathbb{Q}(\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \sqrt{-D})$. For $t \neq (-1 \pm 3\sqrt{-3})/2$, we have $[K : \mathbb{Q}] = 6$ and the minimal polynomial of α over $\mathbb{Q}$ is $f_t\overline{f_t}$.

Theorem 7. *Let $t \in \mathbb{Z}_k \setminus \mathbb{Z}$, $L := \mathbb{Q}(\alpha, \overline{\alpha})$ and let $G := \mathrm{Gal}(L/\mathbb{Q})$. Let τ denote the complex conjugation.*

1. *If $t \in T_1 := \{(-1 \pm 3\sqrt{-3})/2\}$, then $L = \mathbb{Q}(\sqrt{-3})$ and $G = \langle\tau\rangle \simeq C_2$.*
2. *If $t \in T_2 := \{\pm\sqrt{-7}, -1 \pm \sqrt{-7}\}$, then $L = K = \mathbb{Q}(\alpha)$ and $G = \langle\sigma\rangle \simeq C_6$ with $\sigma = (\alpha^{(1)}\overline{\alpha^{(3)}}\alpha^{(2)}\overline{\alpha^{(1)}}\alpha^{(3)}\overline{\alpha^{(2)}})$.*
3. *If*

$$\mathrm{Re}\, t = -1/2 \text{ and } t \notin T_1$$

or if

$$t \in T_3 := \{(-5 \pm \sqrt{-11})/2, (3 \pm \sqrt{-11})/2, (-13 \pm \sqrt{-23})/2, (-3 \pm \sqrt{-23})/2,$$
$$(1 \pm \sqrt{-23})/2, (11 \pm \sqrt{-23})/2, (-7 \pm \sqrt{-31})/2, (5 \pm \sqrt{-31})/2\},$$

then $L = K = \mathbb{Q}(\alpha)$ *and* $G = \langle\sigma,\tau\rangle \simeq S_3$ *with* $\sigma = (\alpha^{(1)}\alpha^{(2)}\alpha^{(3)})(\overline{\alpha^{(1)}\alpha^{(3)}\alpha^{(2)}})$. *There are four fields between* $\mathbb{Q}$ *and* K*, namely the quadratic field* $k = \mathbb{Q}(\sqrt{-D})$ *and three cubic fields. One of the cubic fields is real.*

(4) *If* $\operatorname{Re} t \neq -1/2$ *and* $t \notin T_2 \cup T_3$*, then* $[L : K] = 3$ *and* $G = \langle\sigma,\sigma',\tau\rangle$ *with* $|G| = 18$*,* $\sigma = (\alpha^{(1)}\alpha^{(2)}\alpha^{(3)})$ *and* $\sigma' = (\overline{\alpha^{(1)}\alpha^{(2)}\alpha^{(3)}})$*. Furthermore, the only field between* $\mathbb{Q}$ *and* K *is* k*.*

Proof. We assume $|t| \geq 21$, leaving the remaining cases to Pari [2]. We note that the complex conjugation τ is an element of G.

Assume first that G is cyclic, generated by some σ. Without loss of generality, we may assume that σ^2 is the automorphism which induces (5). It is clear that $\tau = \sigma^3$, which implies $\sigma = (\alpha^{(1)}\overline{\alpha^{(3)}}\alpha^{(2)}\overline{\alpha^{(1)}}\alpha^{(3)}\overline{\alpha^{(2)}})$. Let

$$u_1 := \left(4 - \prod_{j=1}^{3}(\alpha^{(j)} + \overline{\alpha^{(j)}})\right)^2 - 2\prod_{j=1}^{3}\left((\alpha^{(j)})^2 + (\overline{\alpha^{(j)}})^2\right).$$

Since u_1 is invariant under σ and $u_1 \in \mathbb{Z}_L$, we have $u_1 \in \mathbb{Z}$. Using (7), we get $|u_1| < 1$ for $|t| \geq 21$. This implies $u_1 = 0$. Eliminating α and $\overline{\alpha}$ from the set of equations $u_1 = 0$, $f(\alpha) = 0$ and $\overline{f(\alpha)} = 0$, we get $D = 0$ or $\operatorname{Im} t = 0$, which has been excluded.

Next, we assume that $L = K$ but G is not cyclic. Let $\sigma \in G$ be the automorphism which maps $\alpha^{(1)}$ to $\alpha^{(2)}$. Since $\sigma(\overline{\alpha^{(1)}}) \neq \overline{\alpha^{(1)}}$ because $|G| = 6$, the only possibility for σ is $\sigma = (\alpha^{(1)}\alpha^{(2)}\alpha^{(3)})(\overline{\alpha^{(1)}\alpha^{(3)}\alpha^{(2)}})$. We note that $\tau\sigma = \sigma^2\tau$ and get $G = \langle\sigma,\tau\rangle$. We consider $\theta_j := \sigma^j(\alpha + \overline{\alpha})$, $j = 1, 2, 3$, and

$$u_2 := 1 - \theta_1\theta_2\theta_3 - \theta_1\theta_2 - \theta_2\theta_3 - \theta_3\theta_1 - t - \overline{t}.$$

As above, we obtain $u_2 = 0$ for $|t| \geq 20$. This yields $\operatorname{Re} t = -1/2$.

Conversely, if $\operatorname{Re} t = -1/2$, we get $\overline{f_t}(-1-\alpha) = 0$ and we obtain the claimed Galois group.

Finally, $\overline{\alpha} \notin K$ yields $[L : K] = 18$ and the asserted Galois group.

The assertions on the fields between $\mathbb{Q}$ and K follow from Galois Theory. □

For the case $\operatorname{Re} t = -1/2$, we will need the following relation.

Lemma 8. *For* $\operatorname{Re} t = -1/2$*,* $|t| \geq 6$*, and* $\alpha = \alpha^{(1)}$ *as calculated in Lemma 4, we have*

$$\overline{\alpha^{(1)}} = -1 - \alpha^{(1)}.$$

Proof. Using (6), we get

$$\left|\overline{\alpha^{(1)}} - (-1 - \alpha^{(1)})\right| = \left|\frac{3 + 10t + 10t^2}{(t + t^2)^3} + L_6\left(\frac{10}{|t|^{7/2}}\right)\right| \leq 0.86.$$

In the proof of Theorem 7, we noticed that $-1 - \alpha^{(1)}$ is a root of $\overline{f_t}$. From (8) we see that $|\overline{\alpha^{(i)}} - \overline{\alpha^{(1)}}| > 1$ for $i = 2, 3$. This yields the assertion. □

5. Unit group

Let $(x, y) \in \mathbb{Z}_k$ be a solution to (1). From

$$\prod_{l=1}^{3}(x - \alpha^{(l)}y) = F_t(x, y) = \mu \tag{10}$$

we conclude that $\beta^{(l)} := x - \alpha^{(l)}y$, $l = 1, 2, 3$, are units in $\mathfrak{O} := \mathbb{Z}_k[\alpha]$.

Therefore, we investigate the unit structure of $\mathfrak{O}$. It can easily be checked that

$$1 = \alpha(\alpha^2 - (t-1)\alpha - (t+2)) = (\alpha + 1)(2 + t\alpha - \alpha^2),$$

which implies that α and $(\alpha + 1)$ are units in $\mathfrak{O}$.

Lemma 9. *Let $|t| \geq 20$ and*
$$\zeta = \begin{cases} (1+\sqrt{-3})/2 & \text{if } D = 3, \\ i & \text{if } D = 1, \\ -1 & \text{otherwise.} \end{cases}$$
Then the index I of $\langle \zeta, \alpha, \alpha+1 \rangle$ in the unit group $\mathfrak{O}^\times$ can be bounded by
$$I \leq 5.03 \log^2 |t| \,. \tag{11}$$

Proof. Assume that $\rho \in \mathfrak{O}$ is a root of unity. Since $\mathbb{Q}(\rho)$ is Galois and since by Theorem 7, the only Galois subfield between $\mathbb{Q}$ and K is k, we see that $\rho \in k$. It is clear that the torsion group of $\mathbb{Z}_k^\times$ (and therefore of $\mathfrak{O}^\times$) is generated by ζ.

Using (7), we calculate
$$0.97 \log^2 |t| \leq -\det \begin{pmatrix} \log|\alpha^{(1)}| & \log|\alpha^{(1)}+1| \\ \log|\alpha^{(2)}| & \log|\alpha^{(2)}+1| \end{pmatrix} \leq 1.03 \log^2 |t| \,. \tag{12}$$
This implies that α and $\alpha+1$ are independent units of $\mathfrak{O}$.

By Friedman [7, Theorem B], the regulator of K can be bounded by $\operatorname{Reg} \mathbb{Z}_K \geq 0.2052$. From Pohst and Zassenhaus [16, p. 361], we conclude that
$$I = [\mathfrak{O}^\times : \langle \zeta, \alpha, \alpha+1 \rangle] \leq [\mathbb{Z}_K^\times : \langle \zeta, \alpha, \alpha+1 \rangle] = \frac{\operatorname{Reg} \langle \zeta, \alpha, \alpha+1 \rangle}{\operatorname{Reg} \mathbb{Z}_K} \leq \frac{1.03 \log^2 |t|}{0.2052} \leq 5.03 \log^2 |t| \,.$$
□

6. Approximation Properties

We call a solution (x, y) to (1) a solution of *type* j, if
$$|\beta^{(j)}| = \min_{l=1,2,3} |\beta^{(l)}|.$$

From
$$|y||\alpha^{(l)} - \alpha^{(j)}| \leq |x - \alpha^{(l)} y| + |x - \alpha^{(j)} y| \leq 2|\beta^{(l)}|,$$
and (10) we conclude that
$$|\beta^{(j)}| = \frac{1}{\prod_{l \neq j} |\beta^{(l)}|} \leq \frac{4}{\prod_{l \neq j} \left|\alpha^{(l)} - \alpha^{(j)}\right|} \cdot \frac{1}{|y|^2} = \frac{4}{|f_t'(\alpha^{(j)})|} \cdot \frac{1}{|y|^2}. \tag{13}$$

Lemma 10. *Let $|t| \geq 20$ and (x, y) be a solution to (1) of type j. Then $\bigl(-(x+y), x\bigr)$ is a solution to (1) of type $j' := (j+1) \bmod 3$.*

Proof. We may assume $|y| \geq \sqrt{5}$ by Lemma 5. By (4), $\bigl(-(x+y), x\bigr)$ is also a solution to (1). From (5) and (13) we see that
$$\left| \frac{-(x+y)}{x} - \alpha^{(j')} \right| = \left| \frac{\alpha^{(j)} - x/y}{\alpha^{(j)} \cdot x/y} \right| \leq \frac{4}{|y|^3 \cdot |f_t'(\alpha^{(j)})|} \cdot \frac{1}{|\alpha^{(j)}|} \cdot \frac{1}{|\alpha^{(j)}| - 4/(|y|^3 \cdot |f_t'(\alpha^{(j)})|)}.$$

From (7), the assumption $|y| \geq \sqrt{5}$, and (13) we conclude that
$$\min_l \left| \frac{-(x+y)}{x} - \alpha^{(l)} \right| = \left| \frac{-(x+y)}{x} - \alpha^{(j')} \right|.$$
□

Lemma 10 shows that if there is a solution to (1) which is not listed in Table 1, then there is also a solution (x, y) of type 1 which is not contained in Table 1.

7. Stable Growth

Let (x,y) be a solution to (1) of type 1 which is not listed in Table 1 and $|t|\geq 20$. By Lemma 5, we may assume $|y|\geq\sqrt{5}$. For $l=2,\,3$, we have

$$|\beta^{(l)}| = |y|\cdot|\alpha^{(1)}-\alpha^{(l)}|\cdot\left|1+\frac{x/y-\alpha^{(1)}}{\alpha^{(1)}-\alpha^{(l)}}\right|. \tag{14}$$

Taking logarithms and using (13) and (7), we get

$$\log|\beta^{(l)}| = \log|y| + \log|\alpha^{(1)}-\alpha^{(l)}| + L_{20}\left(\frac{c_l}{|t|^3\,|y|^3}\right) \tag{15}$$

for $c_2=4.6$ and $c_3=4.37$. Taking this for $l=3$ and $l=2$ and eliminating $\log|y|$, we get

$$|\Lambda| := \left|\log\left|\frac{\beta^{(2)}}{\beta^{(3)}}\right| - \log\left|\frac{\alpha^{(1)}-\alpha^{(2)}}{\alpha^{(1)}-\alpha^{(3)}}\right|\right| \leq \frac{8.97}{|t|^3\,|y|^3}. \tag{16}$$

From (14) we see that $\Lambda=0$ would imply

$$\frac{x/y-\alpha^{(1)}}{\alpha^{(1)}-\alpha^{(2)}} = \frac{x/y-\alpha^{(1)}}{\alpha^{(1)}-\alpha^{(3)}}.$$

This would lead to $x/y=\alpha^{(1)}$ or $\alpha^{(3)}=\alpha^{(2)}$, which are both contradictions. Therefore we have $\Lambda\neq 0$.

Lemma 9 implies that $\left(\beta^{(l)}\right)^I = \zeta^{u_0}\left(\alpha^{(l)}\right)^{u_1}\left(\alpha^{(l)}+1\right)^{u_2}$ for some integers u_0, u_1, and u_2. Taking logarithms yields

$$\log|\beta^{(l)}| = \frac{u_1}{I}\log|\alpha^{(l)}| + \frac{u_2}{I}\log|\alpha^{(l)}+1|, \qquad l=1,2,3. \tag{17}$$

From (16), we obtain

$$|\Lambda| = \left|\frac{u_1}{I}\log\left|\frac{\alpha^{(2)}}{\alpha^{(3)}}\right| + \frac{u_2}{I}\log\left|\frac{\alpha^{(2)}+1}{\alpha^{(3)}+1}\right| - \log\left|\frac{\alpha^{(1)}-\alpha^{(2)}}{\alpha^{(1)}-\alpha^{(3)}}\right|\right| \leq \frac{8.97}{|t|^3\,|y|^3}.$$

Taking advantage of (5), this linear form can be rewritten as a linear form in two logarithms:

$$|I\cdot\Lambda| = \left|v_1\log|\alpha| + v_2\log\left|1+\frac{1}{\alpha}\right|\right| \leq \frac{8.97\cdot I}{|t|^3\,|y|^3}, \tag{18}$$

where

$$v_1 = u_1-u_2 \qquad\text{and}\qquad v_2 = 2u_1+u_2-I$$

are integers.

If $\operatorname{Re} t=-1/2$, we have

$$\left|1+\frac{1}{\alpha}\right|^2 = 1+\frac{1}{\alpha}+\frac{1}{\overline{\alpha}}+\frac{1}{\alpha\overline{\alpha}} = 1+\frac{\alpha+\overline{\alpha}+1}{\alpha\overline{\alpha}} = 1$$

by Lemma 8. This implies $\log\left|1+\alpha^{-1}\right|=0$. Therefore, (18), (11), and (7) yield $|v_1|<1$ for $|t|\geq 20$. This is a contradiction to $\Lambda\neq 0$. Hence, there are no solutions which are not listed in Table 1 for $\operatorname{Re} t=-1/2$ and $|t|\geq 20$. This proves Theorem 3 for $|t|\geq 20$.

It is now crucial to observe that $\log|\alpha|\sim\log|t|$, whereas $\log\left|1+\alpha^{-1}\right| = O(|t|^{-1})$. This means that v_2 has to be much bigger than v_1.

Assume that $v_1=0$. Using (7), we get

$$\log\left|1+\alpha^{-1}\right| = \frac{1}{2}\left(\frac{1}{t}+\frac{1}{\overline{t}}\right) - \frac{1}{4}\left(\frac{1}{t^2}+\frac{1}{\overline{t}^2}\right) + L_{20}\left(\frac{93}{1000\,|t|^2}\right).$$

Simple estimates show that this implies

$$\left|\log\left|1+\alpha^{-1}\right|\right| \geq \frac{0.4}{|t|^2} \tag{19}$$

for $\operatorname{Re} t \neq -1/2$. Combining this with (18), (11), and the assumption $v_1 = 0$ yields

$$|v_2| \leq \frac{113 \log^2 |t|}{|y|^3 |t|}. \tag{20}$$

For $|t| \geq 351$, this bound is smaller than 1. This yields a contradiction to $\Lambda \neq 0$.

Therefore, we get $v_1 \neq 0$. This implies $|v_1| \geq 1$ and from (18), (19), and (7), we obtain

$$|v_2| \geq \frac{1}{\log|1+\alpha^{-1}|} \left(|v_1 \log|\alpha|| - |\Lambda|\right) \geq \frac{\log|\alpha|}{\log|1+\alpha^{-1}|} - \frac{8.97 \cdot I}{0.4 |y|^3 |t|} \geq 0.89 |t| \log|t|.$$

We solve the system (17) of linear equations in u_1/I and u_2/I by Cramer's rule and using (15). We obtain

$$-\frac{v_2}{I} = \left(\frac{3}{\log|t|} + L_{20}\left(\frac{2.93}{|t| \log|t|} \right) \right) \log|y| + L_{20}(4.3).$$

Combining this with (20) and (11) yields

$$\log|y| \geq \log y_0 := 5.6 \cdot 10^{-2} \cdot |t| - 1.37 \cdot \log|t|. \tag{21}$$

We have proved the property which is called "stable growth" by E. Thomas [19]:

Proposition 11. *Let $|t| \geq 351$ and x, $y \in \mathbb{Z}_{\mathbb{Q}(t)}$ be a solution to* (1) *which is not listed in Theorem 1. Then*

$$\max\{|x|, |y|, |x+y|\} \geq \frac{1.057^{|t|}}{|t|^{1.37}}.$$

8. Lower Bound for $|\Lambda|$

We use the following refinement, due to Mignotte [15], of a theorem of Laurent, Mignotte, and Nesterenko [13] on linear forms in two logarithms. For any non-zero algebraic number γ of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $c\prod_{j=1}^{d}(X - \gamma^{(j)})$, we denote by

$$h(\gamma) = \frac{1}{d}\left(\log|c| + \sum_{j=1}^{d} \log\max(1, |\gamma^{(j)}|) \right)$$

its absolute logarithmic height.

Lemma 12 (Mignotte [15, Theorem 2])**.** *Consider the linear form*

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1,$$

where b_1 and b_2 are positive integers, α_1 and α_2 are non-zero algebraic numbers of absolute value at least 1, *and $\log \alpha_1$ and $\log \alpha_2$ are any values of their logarithms.*

Suppose that α_1 and α_2 are multiplicatively independent. Put

$$D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]/[\mathbb{R}(\alpha_1, \alpha_2) : \mathbb{R}].$$

Let a_1, a_2, h, k be real positive numbers, and ρ a real number > 1. Put $\lambda = \log \rho$, $\chi = h/\lambda$ and suppose that $\chi \geq \chi_0$ for some number $\chi_0 \geq 0$ and that

$$\begin{aligned} h &\geq D\left(\log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1} \right) + \log \lambda + f(\lceil K_0 \rceil) \right) + 0.023, \\ a_i &\geq \max\{1, \rho|\log \alpha_i| - \log|\alpha_i| + 2Dh(\alpha_i)\}, \quad (i = 1, 2) \\ a_1 \cdot a_2 &\geq \lambda^2, \end{aligned}$$

where

$$f(x) = \log \frac{(1 + \sqrt{x-1})\sqrt{x}}{x-1} + \frac{\log x}{6x(x-1)} + \frac{3}{2} + \log \frac{3}{4} + \frac{\log \frac{x}{x-1}}{x-1}$$

and

$$K_0 = \frac{1}{\lambda}\left(\frac{\sqrt{2+2\chi_0}}{3} + \sqrt{\frac{2(1+\chi_0)}{9} + \frac{2\lambda}{3}\left(\frac{1}{a_1} + \frac{1}{a_2} \right) + \frac{4\lambda\sqrt{2+\chi_0}}{3\sqrt{a_1 a_2}}} \right)^2 a_1 a_2.$$

Put

$$v = 4\chi + 4 + 1/\chi, \qquad m = \max\{2^{5/2}(1+\chi)^{3/2}, (1+2\chi)^{5/2}/\chi\}.$$

Then we have the lower bound

$$\begin{aligned}\log|\Lambda| \geq &-\frac{1}{\lambda}\left(\frac{v}{6} + \frac{1}{2}\sqrt{\frac{v^2}{9} + \frac{4\lambda v}{3}\left(\frac{1}{a_1} + \frac{1}{a_2}\right) + \frac{8\lambda m}{3\sqrt{a_1 a_2}}}\right)^2 a_1 a_2 \\ &- \max\big\{\lambda(1.5+2\chi) + \log(((2+2\chi)^{3/2} + (2+2\chi)^2\sqrt{k^*})A + (2+2\chi)), D\log 2\big\},\end{aligned}$$

where

$$A = \max\{a_1, a_2\} \text{ and } k^* = \frac{1}{\lambda^2}\left(\frac{1+2\chi}{3\chi}\right)^2 + \frac{1}{\lambda}\left(\frac{2}{3\chi} + \frac{2}{3}\cdot\frac{(1+2\chi)^{1/2}}{\chi}\right).$$

Let $\sigma := \operatorname{sign}(|\alpha^{(2)}| - 1)$. We will apply Lemma 12 to the linear form in logarithms $\sigma 2I\Lambda$, which can be written as

$$\sigma 2I\Lambda = \sigma v_1 \log(\alpha^{(1)}\overline{\alpha^{(1)}}) - (-v_2)\log(\alpha^{(2)}\overline{\alpha^{(2)}})^\sigma$$

by (18) and (5). We note that $\log(\alpha^{(1)}\overline{\alpha^{(1)}})$ and $\log(\alpha^{(2)}\overline{\alpha^{(2)}})$ are linearly independent over $\mathbb{Q}$ by (12).

From now on, we will assume $|t| \geq 10^9$ in order to obtain sharper bounds.

We have

$$\begin{aligned}D &= 9 \\ h(\alpha^{(1)}\overline{\alpha^{(1)}}) &= h(\alpha^{(2)}\overline{\alpha^{(2)}}) \\ &\leq \frac{2}{3}\left(\left|\log|\alpha^{(1)}|\right| + \left|\log|\alpha^{(2)}|\right|\right) \leq \frac{2}{3}\log|t| + \frac{1}{3t} + \frac{1}{3\overline{t}} + L_{10^9}\left(4.87\cdot 10^{-2}\cdot\frac{\log|t|}{|t|^2}\right).\end{aligned}$$

We choose $\rho = 30.0658$ and obtain

$$\begin{aligned}a_1 &= 12\log|t| + 6\left(\frac{1}{t} + \frac{1}{\overline{t}}\right) + \frac{62.76}{|t|} + 0.877\cdot\frac{\log|t|}{|t|^2}, \\ a_2 &= 74.14\cdot\log|t| + 6\left(\frac{1}{t} + \frac{1}{\overline{t}}\right) + 7.31\cdot\frac{\log|t|}{|t|^2}, \\ 0 < \sigma v_1 &\leq \frac{14.7}{|t|}\log|y| + 19.5\cdot\frac{\log|t|}{|t|}, \\ -v_2 &\leq \left(\frac{2500}{171}\log|t| + 1.46\cdot\frac{\log|t|}{|t|}\right)\log|y| + \left(19.50\log^2|t| + 2.42\cdot\frac{\log^2|t|}{|t|}\right), \\ h &= 9\log\log|y| + \left(7.519 + 177.2\cdot\frac{\log|t|}{|t|} + \frac{3.72}{|t|}\right), \\ K_0 &\geq 4\,862\,274.\end{aligned}$$

Since the only available asymptotic information on $|y|$ is (21), we use this bound everywhere but in the asymptotic main term. After some calculations, we finally get

$$\log|2I\Lambda| \geq -\log^2\log|y|\cdot(3073\log^2|t| + 7505\log|t| + 941.1\sqrt{\log|t|} + 44510). \tag{22}$$

We consider the function

$$\begin{aligned}h(y,t) := 3\log|y| - \log^2\log|y|\cdot(3073\log^2|t| + 7505\log|t| + 941.1\sqrt{\log|t|} + 44510) \\ + 3\log|t| - 2\log\log|t| - 4.384.\end{aligned}$$

By (22), (18), and (11), we have

$$h(y,t) \leq 0 \tag{23}$$

for $|t| \geq 10^9$.

Since

$$\frac{\partial}{\partial\log|y|}h(y,t) = 3 - 2(3073\log^2|t| + 7505\log|t| + 941.1\sqrt{\log|t|} + 44510)\frac{\log\log|y|}{\log|y|} > 0$$

for $|y| \geq y_0$ (y_0 has been defined in (21)) and $|t| \geq 2.15 \cdot 10^8$, we have $h(y,t) \geq h(y_0,t)$. But $h(y_0,t) > 0$ for $|t| \geq 3\,023\,195\,238$, which is a contradiction to (23). This proves Theorem 1.

9. Proof of Theorem 3

Theorem 3 has already been proved for $|t| \geq 20$ in Section 7. For the remaining small values of $|t|$, the asymptotic expansions do not help. Especially, we have to redefine the order of the roots of f_t, since $\alpha^{(1)}$ was defined via an asymptotic expansion.

We easily calculate

$$\hat{f}_{t_2}(x_2) := f_{-1/2+\mathrm{i}t_2}(-1/2+\mathrm{i}x_2) = -x_2^3 + t_2 x_2^2 - \frac{9}{4}x_2 + \frac{t_2}{4}.$$

Since $\operatorname{discr} \hat{f}_{t_2} < 0$ for $t_2 \neq \pm 3\sqrt{3}/2$, we conclude that there is exactly one root of $f_{-1/2+\mathrm{i}t_2}$ with real part equal to $-1/2$. This root will be denoted by $\alpha := \alpha^{(1)}$. Since $\alpha^{(1)} + \overline{\alpha^{(1)}} = -1$, we have

$$|\alpha^{(1)} + 1| = |\alpha^{(1)}|.$$

The relations (5) give us the other roots of f_t.

The type of a solution is defined as in Section 6. However, we cannot use Lemma 10.

For each $1 \leq |t| \leq 20$ with $\operatorname{Re} t = -1/2$ and for each pair (j,l) with $1 \leq l \neq j \leq 3$, we calculate α explicitly and set

$$\hat{c}_{tjl} := \left| \frac{4}{f_t'(\alpha^{(j)})} \cdot \frac{1}{\alpha^{(j)} - \alpha^{(l)}} \cdot \frac{1}{11^{3/2}} \right|, \qquad c_{tjl} := \max\{\log(1+\hat{c}_{tjl}), -\log(1-\hat{c}_{tjl})\}.$$

By (13) and (14), we get

$$\log|\beta^{(l)}| = \log|y| + \log|\alpha^{(j)} - \alpha^{(l)}| + \varepsilon_{tjl}$$

for some $|\varepsilon_{tjl}| \leq c_{tjl}$ and for all solutions (x,y) to (1) of type j with $|y| \geq \sqrt{11}$.

As in Section 7, we obtain

$$|I \cdot \Lambda| = \left| v_1 \log|\alpha^{(j)}| + v_2 \log\left|1 + \frac{1}{\alpha^{(j)}}\right| \right| \leq c_{tj} := \frac{\log^2|\alpha|}{0.2052} \cdot \sum_{\substack{l=1 \\ l \neq j}}^{3} c_{tjl}.$$

We note that $\log|1+1/\alpha^{(1)}| = \log|\alpha^{(2)}| = 0$ and $\log|\alpha^{(1)}| = -\log|1+1/\alpha^{(2)}| = -\log|\alpha^{(3)}| = \log|1+1/\alpha^{(3)}| = \log|\alpha|$. This yields

$$|w_j \log|\alpha|| \leq c_{tj},$$

where $w_1 = v_1$, $w_2 = -v_2$, and $w_3 = v_2 - v_1$ are integers.

However, it is checked numerically that $c_{tj}/\log|\alpha| < 1$ for all relevant t and j. Therefore, we get $w_j = 0$, which implies $\Lambda = 0$. But this is impossible by the argument in Section 7.

Finally, we have to find all solutions with $|y| < \sqrt{11}$. The procedure described in Lemma 5 exactly yields the solutions listed in Table 2.

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