On a Conjecture of E. Thomas concerning parametrized Thue Equations

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1 Introduction

For an irreducible form $F \in \mathbb{Z}[X, Y]$ of degree at least 3 and a nonzero integer m, the Diophantine equation

$$F(X, Y) = m$$

is called *Thue equation* in honour of A. Thue, who proved in 1909 [19] that the number of its solutions in integers is finite. Thue's result is not effective, but in 1968, A. Baker [1] gave an upper bound for the solutions using his lower bounds for linear forms in logarithms of algebraic numbers. Since then, algorithms for the solution of single Thue equations have been

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developed, see Pethő and Schulenberg [14], Tzanakis and de Weger [20], and Bilu and Hanrot [4].

In 1990, E. Thomas [17] considered for the first time a parametrized family of cubic Thue equations of positive discriminant. In the last decade, several such families of degrees 3 to 6 have been investigated, see [8] for further references. In all these families, there were only two types of solutions: Firstly, there are polynomial solutions $X(a), Y(a) \in \mathbb{Z}[a]$ which satisfy $F(X,Y) = \pm 1$ in $\mathbb{Z}[a]$, and secondly, there may be some further solutions for special (small) values of the parameter.

The family of Thue equations

$$aX^n - bY^n = \pm 1$$

has been considered by several authors, Bennett [3] recently showed that for $ab \neq 0$ and $n \geq 3$ this equation has at most one solution in positive integers (x, y).

A further step is the investigation of classes of parametrized families of arbitrary degree such as

$$(1) \qquad F_a(X, Y) := \prod_{i=1}^{n}(X - p_i(a)Y) - Y^n = \pm 1, \qquad a \in \mathbb{N},$$

where $p_1, \ldots, p_n \in \mathbb{Z}[a]$ are polynomials, which have been called *split families* by E. Thomas [18]. For $i = 1, \ldots, n$ it can be seen easily that $(X, Y) \in \{\pm(p_i, 1), (\pm 1, 0)\}$ are polynomial solutions. Thomas conjectured that if

$$p_1 = 0, \qquad 0 < \deg p_2 < \cdots < \deg p_n$$

and the polynomials are monic, there are no further solutions for sufficiently large values of the parameter a. In [18] he proved this conjecture for $n = 3$ under some technical hypothesis.

Halter-Koch, Lettl, Pethő and Tichy [6] considered (1) for $p_1 = 0, p_2 = d_2, \ldots, p_{n-1} = d_{n-1}$ and $p_n = a$, where $d_2, \ldots, d_{n-1}$ are distinct fixed integers. They found all solutions for sufficiently large values of a assuming a conjecture of Lang and Waldschmidt [10] — which is a very sharp bound for linear forms in logarithms of algebraic numbers —, provided that the corresponding number field is primitive (i. e. $\mathbb{Q}$ is the only proper subfield), which is the case for almost all choices (in the sense of thin sets) of the parameters.

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The first unconditional result on split families of arbitrary degree has been given by the author in [7], where (1) has been considered for $p_1 = -a, p_2 = d_2, \ldots, p_{n-1} = d_{n-1}$ and $p_n = a$, where $d_2, \ldots, d_{n-1}$ are distinct fixed integers. If $\sum_i d_i \neq 0$ or $\prod_i d_i \neq 0$, then the only solutions are polynomial solutions with $|Y| \leq 1$.

In [9], Heuberger and Tichy considered a multivariate version of (1): Let $p_i \in \mathbb{Z}[a_1, \ldots, a_r]$ for some $r \in \mathbb{N}$. Assume

$$\deg p_1 < \cdots < \deg p_{n-2} < \deg p_{n-1} = \deg p_n,$$
$$\text{LH}\,(p_n) = \text{LH}\,(p_{n-1})\,, \text{ but } p_n \neq p_{n-1},$$

where $\text{LH}\,(p)$ denotes the homogeneous part of maximal degree of a polynomial p, and suppose that the polynomials p_i satisfy suitable growth conditions. Then there is a constant t_0 such that for all $a_1, \ldots, a_r$ satisfying $t_0 \leq \min_k a_k$ and $\max_k a_k \leq (\min_k a_k)^\tau$ the Diophantine equation (1) has only polynomial solutions with $|Y| \leq 1$, where $\tau > 1$ is an explicitly given constant (depending on the degrees of the polynomials only).

These two results depend heavily on the symmetric nature of the polynomials p_i. In this paper, we shall prove Thomas' conjecture for a very large class of polynomials p_i, subject only to certain technical conditions on the degrees of the p_i.

After the presentation of the results in this section we shall, in Section 2, give some comments on the technical hypothesis occuring in our Theorems 1 and 2 and we will discuss its relation to Thomas' technical hypothesis in the case $n = 3$. In Sections 3 and 4 we collect and adapt standard material for the solution of parametrized Thue equations. In Section 5 we will present the main idea to exclude "small" solutions, which will be carried out in detail in Sections 6 and 7. In Section 8 we will exclude "large" solutions using Baker's method via application of a result of Bugeaud and Győry [5]. Finally we will prove a weaker formulation of the technical hypothesis in Section 9.

The main result of the present paper is

Theorem 1. *Let* $n \in \mathbb{N}$, $n \geq 4$ *and* $p_i \in \mathbb{Z}[a]$ *be monic polynomials for* $i = 1, \ldots, n$. *We write* $d_i := \deg p_i$,

$$(2) \qquad\qquad e_i := (i-1)d_i + \sum_{l=i+1}^{n} d_l$$

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and allow $p_1 = 0$ *(we write $d_1 = -1$ in this case). Moreover, we define recursively*

(3) $\quad \psi_i := \dfrac{(e_2 + d_2)(d_{i+1} - d_3)}{e_{i+1} + d_{i+1}} + \displaystyle\sum_{h=3}^{i-1} \dfrac{d_{i+1} - d_{h+1}}{e_{i+1} + d_{i+1}} \psi_h, \qquad 3 \le i \le n - 1.$

If $\psi_i \in \mathbb{N}$ for all $3 \le i \le n - 1$, we define for $(j, j') \in \{(1, 2), (2, 1)\}$

$$Q_j^+ := (p_3 - p_j)^{e_1 + d_3} \prod_{k=4}^{n} (p_k - p_3)^{\psi_{k-1}},$$

$$Q_j^- := (p_2 - p_1)^{e_1 + 2d_3 - d_2} (p_3 - p_{j'})^{2(d_3 - d_2)} \prod_{k=4}^{n} (p_k - p_{j'})^{\psi_{k-1} + d_3 - d_2}.$$

Assume

(4) $$d_1 < d_2 < \cdots < d_{n-1} < d_n.$$

If there is a $3 \le k \le n - 1$ such that $\psi_k \notin \mathbb{N}$ or if we have

(5) $$\deg(Q_j^+ - Q_j^-) > \deg Q_j^- - e_1 - d_2$$

for $(j, j') = (1, 2)$ and for $(j, j') = (2, 1)$, then there is a (computable) constant $a_0 = a_0(p_1, \ldots, p_n)$ depending on the coefficients of the polynomials p_i such that for all integers $a \ge a_0$ the Diophantine equation

(6) $$F_a(X, Y) := \prod_{i=1}^{n} (X - p_i(a)Y) - Y^n = \pm 1$$

has only the solutions

(7) $$(\pm 1, 0) \quad and \ \pm (p_i(a), 1), 1 \le i \le n.$$

The case $n = 3$ has been excluded in the formulation of Theorem 1 in order to avoid any ambiguities; it is stated explicitly in the following theorem:

Theorem 2. *Let $p_1, p_2, p_3 \in \mathbb{Z}[a]$ be monic polynomials. We write $d_i := \deg p_i$,*

$$e_i := (i - 1)d_i + \sum_{l=i+1}^{3} d_l$$

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and allow $p_1 = 0$ *(we write* $d_1 = -1$ *in this case). We define for* $(j, j') \in \{(1, 2), (2, 1)\}$

$$Q_j^+ := (p_3 - p_j)^{e_1 + d_3},$$
$$Q_j^- := (p_2 - p_1)^{e_1 + 2d_3 - d_2}(p_3 - p_{j'})^{2(d_3 - d_2)}.$$

Assume

(8) $d_1 < d_2 < d_3$ *and* $d_2 \geq 1.$

If we have

(9) $\deg(Q_j^+ - Q_j^-) > \deg Q_j^- - e_1 - d_2$

for $(j, j') = (1, 2)$ *and for* $(j, j') = (2, 1)$, *then there is a (computable) constant* $a_0 = a_0(p_1, p_2, p_3)$ *depending on the coefficients of the polynomials* p_i *such that for all integers* $a \geq a_0$ *the Diophantine equation*

(10) $F_a(X, Y) := \prod_{i=1}^{3}(X - p_i(a)Y) - Y^n = \pm 1$

has only the solutions

$$(\pm 1, 0) \ and \ \pm (p_i(a), 1), 1 \leq i \leq 3.$$

We remark that if $d_2 = 0$, i. e. $p_1 = 0$, $p_2 = 1$ and $\deg p_3 \geq 1$, the assertion of the theorem is false since $F_a(p_3 - 3, p_3 - 2) = -1$. See however Lee [11], Mignotte and Tzanakis [13], and Mignotte [12], where all solutions of this family are determined.

We also give a weaker formulation of the technical hypothesis:

Corollary 3. *Let* $n \in \mathbb{N}$, $n \geq 3$ *and* $p_i \in \mathbb{Z}[a]$ *be monic polynomials for* $i = 1, \ldots, n$. *We write*

$$p_i(a) = a^{d_i} + c_i a^{d_i - 1} + \ terms \ of \ lower \ degree, \qquad i = 2, \ldots, n,$$

allow $p_1 = 0$ *and assume* (4). *If* $n = 3$, *we assume* (8).
Let
$$\delta_i := \begin{cases} 1 & if \ d_i - d_{i-1} = 1, \\ 0 & otherwise \end{cases}$$

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and

$$e := \sum_{i=2}^{n} d_i.$$

If $\delta_4 = 1$ or

(11)

$$(e-d_2+2d_3)(c_2-\delta_2)+(-e-2d_2+d_3)c_3+(d_3-d_2)\sum_{i=4}^{n} c_i \notin \{2\delta_3, -(e+d_3)\delta_3\},$$

then there is a (computable) constant $a_0 = a_0(p_1, \ldots, p_n)$ depending on the coefficients of the polynomials p_i such that for all integers $a \geq a_0$ the Diophantine equation (6) has only the solutions (7).

If the technical hypothesis in Theorem 1 is checked for small degrees, the following corollary is obtained.

Corollary 4. *Let $n \in \mathbb{N}$, $n \in \{4, 5\}$ and $p_i \in \mathbb{Z}[a]$ be monic polynomials for $i = 1, \ldots, n$, where we allow $p_1 = 0$. If*

(12)
$$\max(\deg p_1, 0) < \deg p_2 < \deg p_3 < \cdots < \deg p_n$$
$$\max(\deg p_1, 0) + \deg p_2 + \ldots + \deg p_n < 15,$$

then there is a (computable) constant $a_0 = a_0(p_1, \ldots, p_n)$ depending on the coefficients of the polynomials p_i such that for all integers $a \geq a_0$ the Diophantine equation

$$F_a(X, Y) = \prod_{i=1}^{n} (X - p_i(a)Y) - Y^n = \pm 1$$

has only the solutions (7).

Note that for $n \geq 6$ condition (12) cannot hold since $\sum \deg p_i \geq 0 + 1 + \cdots + (n-1)$.

2   About the Technical Hypothesis

Obviously, it is easy to check the technical hypothesis in Theorems 1 and 2 for a given set of polynomials p_i. Therefore we shall discuss the problem with given degrees $d_1, \ldots, d_n$ and unknown coefficients of the polynomials

p_i. By the definition of ψ_i in (3) we see that it is rather unlikely that all ψ_i are integers, especially if n is large.

If the ψ_i are indeed all integers, (5) implies that the $\max(0, d_1) + \sum_{i=2}^{n} d_i$ unknown coefficients satisfy a system of $e_1 + d_2 - 1$ algebraic equations. We may further assume that the coefficient of $a^{d_n - 1}$ in $p_n(a)$ vanishes, since we can replace the problem involving $p_i(a)$ by that involving $p_i(a - 1)$.

Hence we have $e_1 + d_2 - 1$ equations in $e_1 + \max(0, d_1) - 1$ unknowns; if we exclude $(d_1, d_2) = (-\infty, 0)$ — which has also been excluded by Thomas in the statement of his conjecture — we have an overdetermined system of algebraic equations. Of course, this system is expected to not even have complex solutions and integer solutions seem to be rather unlikely.

The result of such studies for small degrees is given in Corollary 4, where we excluded again the case $(d_1, d_2) = (-\infty, 0)$.

Proof of Corollary 4. In order to prove Corollary 4 we only have to check the assumptions of Theorem 1. For $n = 5$, there are no degrees $\deg p_i$ such that (12), $\psi_3 \in \mathbb{N}$ and $\psi_4 \in \mathbb{N}$ hold.

For $n = 4$, there are only 13 sets $\{d_1, \ldots, d_4\}$ which satisfy both (12) and $\psi_3 \in \mathbb{N}$. For those degrees we calculate the systems of algebraic equations corresponding to (5) modulo 5 and 7 (using Pari [2]) and calculate a reduced Gröbner basis over $\mathbb{Z}_m$ ($m \in \{5, 7\}$) with respect to total degree ordering of the power products (using the program Gröbner [21]). In all cases, the Gröbner basis consists for at least one of the moduli 5 and 7 of the polynomial 1 only. Therefore the systems of algebraic equations have no integer solutions, which proves (5); thus the assertion follows from Theorem 1. $\quad\square$

In order to compare the technical hypothesis in Theorem 2 with Thomas' technical hypothesis [18], we note that in the case considered in [18], we have $p_1 = 0$ and $\deg p_2 > 0$. Obviously, (9) is equivalent to

$$\log \frac{Q_j^-}{Q_j^+} = \log \left(1 + \frac{Q_j^- - Q_j^+}{Q_j^+} \right) = \Omega \left(\frac{1}{a^{e_1 + d_2 - 1}} \right).$$

We calculate

$$\log \left(\frac{Q_1^-}{Q_1^+} \right) = 3 \log \frac{p_2^{d_3}}{p_3^{d_2}} + 2(d_3 - d_2) \log \left(1 - \frac{p_2}{p_3} \right)$$
$$= 3W(a) - 2(d_3 - d_2) a^{-(d_3 - d_2)} (1 + O(a^{-1})),$$

and

$$\log\left(\frac{Q_2^-}{Q_2^+}\right) = 3\log\frac{p_2^{d_3}}{p_3^{d_2}} - (d_2 + 2d_3)\log\left(1 - \frac{p_2}{p_3}\right)$$
$$= 3W(a) + (d_2 + 2d_3)a^{-(d_3-d_2)}(1 + O(a^{-1})),$$

where

$$1 + P_i(a) = p_i/a^{d_i} \qquad\qquad\qquad i = 2, 3$$
$$W(a) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(d_3 P_2^k - d_2 P_3^k\right) = \sum_{k=1}^{\infty}\frac{w_k}{a^k}$$

is as in [18, (1.9)–(1.11)]. It is easily seen that Thomas' definition of a regular family [18, (1.12)] is equivalent to

$$\log\left(\frac{Q_j^-}{Q_j^+}\right) = \Omega\left(\frac{1}{a^{d_3-d_2}}\right)$$

Since $d_3 - d_2 < e_1 + d_2 - 1 = 2d_2 + d_3 - 1$, this implies (9), therefore Theorem 2 improves Thomas' Theorem 2.

3 Preliminaries

We first consider solutions (x, y) with $|y| \leq 1$.

Lemma 5. *The solutions $(x, y) \in \mathbb{Z}^2$ of (6) with $|y| \leq 1$ are precisely those listed in (7), if a is large enough.*

Proof. The pairs (x, y) listed in (7) are clearly solutions of (6).

Conversely, $y = 0$ implies $x = \pm 1$. If $|y| = 1$, we have $\prod(x - p_i(a)y) = 0$, which yields the solutions listed in (7), or

(13) $$\prod_{i=1}^{n}(x - p_i(a)y) = 2u, \qquad u = \pm 1.$$

By (4) the factors of this product are pairwise distinct for sufficiently large a, consequently for $n \geq 4$, (13) is impossible.

Assume $n = 3$. Then we may have $x - p_1(a)y = k_1, x - p_2(a)y = k_2, x - p_3(a)y = k_3$ for $\{k_1, k_2, k_3\} = \{-1, 1, -2u\}$. This implies $(p_3(a) - p_2(a))y - (k_2 - k_3) = 0$, which has only a finite number of solutions a by (4), thus (13) has no solutions for large a. $\square$

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Consider the polynomial $f_a(X) := F_a(X, 1)$. We will need asymptotic estimates for its roots $\alpha^{(1)}, \dots, \alpha^{(n)}$. Throughout this paper, we will use O-, Ω- and Θ-Notation for $a \to \infty$: Let $f, g : \mathbb{R} \to \mathbb{R}$. If there are constants C and a_0 such that $|f(a)| \leq g(a)$ (resp. $|f(a)| \geq g(a)$) for all $a \geq a_0$, we write $f(a) = O(g(a))$ (resp. $f(a) = \Omega(g(a))$). If both $f(a) = O(g(a))$ and $f(a) = \Omega(g(a))$ hold, then we write $f(a) = \Theta(g(a))$. For brevity, we will write p_i instead of $p_i(a)$ in many situations.

We get a statement which is similar to that of Lemma 5 of [9]:

Lemma 6. *All roots of $f_a(X)$ are real and can be estimated as*

$$\alpha^{(i)} = p_i + \frac{(-1)^{n-i}}{a^{e_i}} + O\left(\frac{1}{a^{e_i+1}}\right), \qquad 1 \leq i \leq n.$$

Proof. We fix some $1 \leq i \leq n$ and set

$$\alpha_{i,M} := p_i + \frac{(-1)^{n-i}}{a^{e_i}}\left(1 + \frac{M}{a}\right),$$

where M is a constant which will be chosen later.

Equation (2) and inequality (4) imply

$$(14) \qquad e_i = (i-1)d_i + \sum_{j=i+1}^{n} d_j = \sum_{j \neq i} \deg(p_i - p_j)$$

and $e_i \geq 3$ for $n \geq 3$.

This yields for $j \neq i$

$$(15) \qquad \alpha_{i,M} - p_j = a^{\deg(p_i - p_j)}(-1)^{\sigma_{ij}}\left(1 + \frac{c_{ij}}{a} + O\left(\frac{1}{a^2}\right)\right),$$

where $c_{ij} \in \mathbb{Z}$ depends on the coefficients of p_i and p_j and

$$\sigma_{ij} = \begin{cases} 0 & \text{if } i > j \\ 1 & \text{if } i < j \end{cases}.$$

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We calculate an asymptotic expansion for $f(\alpha_{i,M})$ using (14) and (15):

$$f(\alpha_{i,M}) = \frac{(-1)^{n-i}}{a^{e_i}}\left(1 + \frac{M}{a}\right)\prod_{j\neq i}(\alpha_{i,M} - p_j) - 1$$

$$= \left(1 + \frac{M}{a}\right)\prod_{j\neq i}\left(1 + \frac{c_{ij}}{a} + O\left(\frac{1}{a^2}\right)\right) - 1$$

$$= \frac{1}{a}\left(M + \sum_{j\neq i}c_{ij}\right) + O\left(\frac{1}{a^2}\right).$$

We can choose constants M_1 and M_2 in such a way that $M_1 + \sum_{j\neq i}c_{ij} > 0$ and $M_2 + \sum_{j\neq i}c_{ij} < 0$, which implies that $f(\alpha_{i,M_1})f(\alpha_{i,M_2}) < 0$ for sufficiently large a. As a result we found a zero of f between α_{i,M_1} and α_{i,M_2} if a is large enough. □

4   Associated Number Field

Since f_a is an irreducible polynomial for sufficiently large a by [7, Proposition 3] and (4), the number field $K := \mathbb{Q}(\alpha)$ generated by one of the roots α of f_a has degree n over $\mathbb{Q}$.

If $(x, y) \in \mathbb{Z}^2$ is a solution of (6), then

(16) $N_{K/\mathbb{Q}}(x - \alpha y) = F(x,y) = \pm 1,$

hence $x - \alpha y$ is a unit in $\mathfrak{O} := \mathbb{Z}[\alpha]$. Therefore, we will describe the structure of the unit group $\mathfrak{O}^\times$.

We define

$$\eta_i := \alpha - p_i.$$

Since $f_a(\alpha) = 0$ we have $\prod_i \eta_i = 1$, which implies that η_i is a unit in $\mathfrak{O}$.

We will show that η_i, $i = 1, \ldots, n-1$, are "sufficiently close" to fundamental units of $\mathfrak{O}^\times$. In order to estimate the index of the group generated by these units, we will need a lower bound for the regulator of the unit group. Since logarithmic terms do not matter for our purposes, we can take any regulator estimate, for instance that of Remak [16].

Lemma 7 (Remak). *Let K be a totally real number field. Then the regulator R_K satisfies*

$$R_K > 0.001.$$

We give an asymptotic description of the $\eta_i^{(k)}$. To simplify notation, we define the abbreviation $l_i^{(k)} := \log \left| \eta_i^{(k)} \right|$.

Lemma 8. *Let* $1 \leq i, k \leq n$, $m := \min(i, k)$ *and* $M := \max(i, k)$. *Then we have*

$$l_i^{(k)} = \log(p_M - p_m) + O\left(\frac{1}{a^{e_m + d_M}}\right), \qquad i \neq k.$$

Moreover, there are $r_{M,m,l} \in \mathbb{Q}$ *for* $l = 0, \ldots, e_1 + d_M - 1$ *depending only on the coefficients of* p_s, $s = 1, \ldots, n$, *such that*

$$l_i^{(k)} = r_{M,m,0} \log a + \sum_{l=1}^{e_1 + d_M - 1} \frac{r_{M,m,l}}{a^l} + O\left(\frac{1}{a^{e_1 + d_M}}\right).$$

In particular we have

$$l_i^{(k)} = \begin{cases} d_M \log a + O\left(\frac{1}{a}\right) & \text{if } i \neq k, \\ -e_i \log a + O\left(\frac{1}{a}\right) & \text{if } i = k. \end{cases}$$

Proof. Assume $i \neq k$. By Lemma 6 we obtain $l_i^{(k)} = \log \left| \alpha^{(k)} - p_i \right| = \log |p_k - p_i + O\left(a^{-e_k}\right)| = \log(p_M - p_m + O\left(a^{-e_m}\right)) = \log[(p_M - p_m)(1 + O\left(a^{-e_m - d_M}\right))] = \log(p_M - p_m) + O\left(a^{-e_m - d_M}\right)$. The power series expansion of $\log(1 + z)$ yields the assertions.

Let now $i = k$. By definition of $\alpha^{(i)}$ we have $l_i^{(i)} = -\sum_{j \neq i} l_j^{(i)}$, and the lemma is proved. $\qquad\square$

Lemma 9. *Let* $\{i_1, \ldots, i_{n-1}\}$ *be a subset of* $\{1, \ldots, n\}$ *of cardinality* $n - 1$ *and*

$$G := \left\langle -1, \eta_{i_1}, \ldots, \eta_{i_{n-1}} \right\rangle \subseteq \mathfrak{O}^\times$$

Then the regulator R_G *can be estimated by*

(17) $$R_G = \Theta\left(\log^{n-1} a\right)$$

and the index $[\mathfrak{O}^\times : G]$ *is bounded by*

(18) $$[\mathfrak{O}^\times : G] = O\left(\log^{n-1} a\right).$$

Proof. Assume first $i_1 = 1, \ldots, i_{n-1} = n - 1$. Then (17) follows from Lemma 8 and [9, Lemma 7]. For arbitrary $i_1, \ldots, i_{n-1}$, the result follows from $l_n^{(i)} = -\sum_{k=1}^{n-1} l_k^{(i)}$.

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Equation (18) is a consequence of Pohst and Zassenhaus [15, p. 361], Lemma 7, and (17):

$$I = [\mathfrak{O}^{\times} : G] = \frac{R_G}{R_{\mathfrak{O}}} \leq \frac{R_G}{R_{\mathfrak{O}_K}} \leq \frac{R_G}{0.001} = O\left(\log^{n-1} a\right).$$

$\square$

5 Approximation Properties of Solutions

Let $(x, y) \in \mathbb{Z}^2$ be a solution of (6) and $\beta := x - \alpha y$. We define the *type j* of a solution (x, y) such that

$$\left|\beta^{(j)}\right| = \min_{i=1,\dots,n} \left|\beta^{(i)}\right|.$$

By [9, Proof of Lemma 12] we obtain

$$\left|\frac{\beta^{(j)}}{y}\right| = O\left(\frac{1}{a^{e_j}}\right),$$

hence for $i \neq j$

$$\left|\frac{\beta^{(i)}}{y}\right| = \left|\frac{x}{y} - \alpha^{(j)} + \alpha^{(j)} - p_j + p_j - \alpha^{(i)}\right| = \left|\eta_j^{(i)}\right| \left(1 + O\left(\frac{1}{a^{e_j+d_2}}\right)\right),$$

and therefore

$$(19) \qquad \log\left|\beta^{(i)}\right| = \log|y| + l_j^{(i)} + O\left(\frac{1}{a^{e_j+d_2}}\right), \qquad i \neq j.$$

The crucial part of the proof of Theorems 1 and 2 will be the following proposition, which shows that only solutions with $|y| \leq 1$ or very large $|y|$ can exist. The latter can be excluded easily by standard methods (see Section 8).

Proposition 10. *Let $(x, y) \in \mathbb{Z}^2$ be a solution of (6) with $|y| \geq 2$. Then*

$$(20) \qquad \log|y| = \Omega\left(\frac{a}{\log^{n-3} a}\right).$$

This proposition will be proved as follows: First we give the parts of the proof which are independent of the type j of the solution, whereas the estimates which depend on the type of the solution will be provided in the following sections.

Since β is a unit by (16), Lemma 9 yields

$$\beta^I = \pm \eta_{i_1}^{u_{i_1}} \ldots \eta_{i_{n-1}}^{u_{i_{n-1}}},$$

where $\{i_1, \ldots, i_{n-1}\}$ is a subset of $\{1, \ldots, n\}$ of cardinality $n-1$, which will be chosen depending on the case j of the solution, $u_{i_1}, \ldots, u_{i_{n-1}}$ are integers and $I = O\left(\log^{n-1} a\right)$.

Taking logarithms of the conjugates $h \in \{1, \ldots, n\} \setminus \{j\}$, we obtain a system of linear equations for the u_{i_k}/I:

$$\log\left|\beta^{(h)}\right| = \frac{u_{i_1}}{I} l_{i_1}^{(h)} + \cdots + \frac{u_{i_{n-1}}}{I} l_{i_{n-1}}^{(h)}, \qquad h \neq j.$$

Cramer's rule yields

$$R\frac{u_{i_k}}{I} = \begin{vmatrix} l_{i_1}^{(1)} & \cdots & l_{i_{k-1}}^{(1)} & \log\left|\beta^{(1)}\right| & l_{i_{k+1}}^{(1)} & \cdots & l_{i_{n-1}}^{(1)} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{i_1}^{(n)} & \cdots & l_{i_{k-1}}^{(n)} & \log\left|\beta^{(n)}\right| & l_{i_{k+1}}^{(n)} & \cdots & l_{i_{n-1}}^{(n)} \end{vmatrix},$$

where the j-th row is omitted and R denotes the determinant of the system matrix, which is (up to a sign) the regulator R_G estimated in Lemma 9.

Applying (19) we obtain

$$(21) \qquad R\frac{u_{i_k}}{I} = M_{j,i_k} \log|y| + \Delta_{j,k} R + O\left(\frac{\log^{n-2} a}{a^{e_j + d_2}}\right),$$

where $\Delta_{j,k} = \pm 1$ if $j \notin \{i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_{n-1}\}$ and 0 else and

$$M_{j,i_k} = \begin{vmatrix} l_{i_1}^{(1)} & \cdots & l_{i_{k-1}}^{(1)} & 1 & l_{i_{k+1}}^{(1)} & \cdots & l_{i_{n-1}}^{(1)} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{i_1}^{(n)} & \cdots & l_{i_{k-1}}^{(n)} & 1 & l_{i_{k+1}}^{(n)} & \cdots & l_{i_{n-1}}^{(n)} \end{vmatrix},$$

where the j-th row is omitted.

By Lemma 8 we see that

$$M_{j,i_k} = G_{j,i_k} \log^{n-2} a + O\left(\frac{\log^{n-3} a}{a}\right)$$

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for some integers G_{j,i_k}.

We will choose a suitable $\mathbb{Z}$-linear combination v_j of u_{i_k}, $k = 1, \ldots, n-1$, and I depending on the type j of the solution such that (21) yields

$$
(22) \qquad R\frac{v_j}{I} = M_j \log |y| + O\left(\frac{\log^{n-2} a}{a^{e_j + d_2}}\right)
$$

with

$$
(23) \qquad M_j = O\left(\frac{\log^{n-3} a}{a}\right).
$$

In the next sections we will prove that our choice of the linear combination v_j implies

$$
(24) \qquad M_j = \Omega\left(\frac{1}{a^{e_j + d_2 - 1}}\right).
$$

If (24) holds, it is clear from (22) that for sufficiently large values of a, v_j does not vanish, since we assume $|y| \geq 2$. If $Rv_j/I > 0$, then $Rv_j/I \geq |R|/I$ because v_j is an integer. Then by (22), (17), and (18) we obtain

$$
M_j \log |y| \geq \frac{|R|}{I} - O\left(\frac{\log^{n-2} a}{a^{e_j + d_2}}\right) = \Omega(1),
$$

which implies $\log |y| = \Omega\left(a/\log^{n-3} a\right)$ by (23). If $Rv_j/I < 0$, an analogous argument holds. Therefore, in order to prove Proposition 10 it suffices to show (23) and (24) for a suitable v_j.

6 Cases $3 \leq j \leq n$

We choose $(i_1, \ldots, i_{n-1}) = (1, 3, \ldots, n)$ and $v_j := u_1$. Our task is to prove (23) and (24) for

$$
M_j = \begin{vmatrix} 1 & l_3^{(1)} & \ldots & l_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & l_3^{(n)} & \ldots & l_n^{(n)} \end{vmatrix},
$$

where the j-th row is omitted. We note that we will not use the technical hypothesis (5) in this section.

We subtract the second from the first row and obtain

$$
(25) \qquad M_j = \begin{vmatrix} 0 & l_3^{(1)} - l_3^{(2)} & \dots & l_n^{(1)} - l_n^{(2)} \\ 1 & l_3^{(2)} & \dots & l_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & l_3^{(n)} & \dots & l_n^{(n)} \end{vmatrix},
$$

where the j-th row is omitted. For $3 \le k \le n$, we have

$$
(26) \qquad \begin{aligned} l_k^{(1)} - l_k^{(2)} &= \log \left| 1 - \left(1 - \frac{\alpha^{(1)} - p_k}{\alpha^{(2)} - p_k} \right) \right| = \log \left(1 + \frac{\alpha^{(2)} - \alpha^{(1)}}{p_k - \alpha^{(2)}} \right) \\ &= \frac{1}{a^{d_k - d_2}} \left(1 + O \left(\frac{1}{a} \right) \right), \end{aligned}
$$

hence $l_k^{(1)} - l_k^{(2)} = O(1/a)$ and (23) is proved.

By Lemma 8, we obtain

$$
(27) \qquad M_j = Q_j(\log a, 1/a) + O \left(\frac{\log^{n-3} a}{a^{e_1 + d_2}} \right),
$$

where $Q_j(\log a, 1/a)$ is some polynomial in $\log a$ and $1/a$ with rational coefficients depending on the coefficients of the polynomials p_i.

We consider the coefficient of $\log^{n-3} a / a^{d_j - d_2}$ in Q_j. We expand the determinant in (25) according to the first row and get

$$
M_j = \sum_{k=3}^{n} (-1)^k (l_k^{(1)} - l_k^{(2)}) N_{jk} = \sum_{k=3}^{n} \frac{(-1)^k}{a^{d_k - d_2}} \left(1 + O \left(\frac{1}{a} \right) \right) N_{jk},
$$

where N_{jk} are the corresponding minors. By (26) and (4), there is no

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contribution of the summands with $k > j$. For $k < j$, we have

$$\frac{N_{jk}}{\log^{n-3} a} =$$

$$\begin{vmatrix} 1 & d_3 & \ldots & d_{k-1} & d_{k+1} & \ldots & d_{j-1} & d_j & d_{j+1} & \ldots & d_n \\ 1 & -e_3 & \ldots & d_{k-1} & d_{k+1} & \ldots & d_{j-1} & d_j & d_{j+1} & \ldots & d_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d_{k-1} & \ldots & -e_{k-1} & d_{k+1} & \ldots & d_{j-1} & d_j & d_{j+1} & \ldots & d_n \\ 1 & d_k & \ldots & d_k & d_{k+1} & \ldots & d_{j-1} & d_j & d_{j+1} & \ldots & d_n \\ 1 & d_{k+1} & \ldots & d_{k+1} & -e_{k+1} & \ldots & d_{j-1} & d_j & d_{j+1} & \ldots & d_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d_{j-1} & \ldots & d_{j-1} & d_{j-1} & \ldots & -e_{j-1} & d_j & d_{j+1} & \ldots & d_n \\ 1 & d_{j+1} & \ldots & d_{j+1} & d_{j+1} & \ldots & d_{j+1} & d_{j+1} & -e_{j+1} & \ldots & d_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d_n & \ldots & d_n & d_n & \ldots & d_n & d_n & d_n & \ldots & -e_n \end{vmatrix}$$

$$+ O\left(\frac{1}{a \log a}\right).$$

Subtracting d_l times the first column from the column corresponding to l, we obtain with $h_l = -e_l - d_l$ $(3 \leq l \leq n)$

$$N_{jk} =$$

$$\log^{n-3} a \begin{vmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 1 & h_3 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & * & \ldots & h_{k-1} & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 1 & * & \ldots & * & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 1 & * & \ldots & * & h_{k+1} & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & * & \ldots & * & * & \ldots & h_{j-1} & 0 & 0 & \ldots & 0 \\ 1 & * & \ldots & * & * & \ldots & * & * & h_{j+1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & * & \ldots & * & * & \ldots & * & * & * & \ldots & h_n \end{vmatrix}$$

$$+ O\left(\frac{\log^{n-4} a}{a}\right),$$

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where $*$ stands for constants which are not important to us. Hence for $k < j$, $N_{jk} = O\left(\log^{n-4} a/a\right)$, and there is no contribution to the term considered. In the same fashion we get for $k = j$

$$
N_{jj} = \log^{n-3} a
\begin{vmatrix}
1 & 0 & \dots & 0 & 0 & \dots & 0 \\
1 & h_3 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & * & \dots & h_{j-1} & 0 & \dots & 0 \\
1 & * & \dots & * & h_{j+1} & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & * & \dots & * & * & \dots & h_n
\end{vmatrix}
+ O\left(\frac{\log^{n-4} a}{a}\right)
$$

$$
= \log^{n-3} a \prod_{\substack{l=3 \\ l \neq j}}^{n} (-e_l - d_l) + O\left(\frac{\log^{n-4} a}{a}\right).
$$

Thus the coefficient of $\log^{n-3} a/a^{d_j - d_2}$ in (27) does not vanish, which implies (24) since $d_j - d_2 < e_1 + d_2$ by (4).

7 Cases $j = 1$ and $j = 2$

In both cases, we choose $(i_1, \dots, i_{n-1}) = (1, 2, 4, \dots, n)$. We set

$$
j' := \begin{cases} 2 & \text{if } j = 1, \\ 1 & \text{if } j = 2 \end{cases}
$$

and get

$$
M_{j,j} = (-1)^{j'}
\begin{vmatrix}
1 & l_{j'}^{(j')} & l_4^{(j')} & \dots & l_n^{(j')} \\
1 & l_{j'}^{(3)} & l_4^{(3)} & \dots & l_n^{(3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & l_{j'}^{(n)} & l_4^{(n)} & \dots & l_n^{(n)}
\end{vmatrix},
$$

$$
M_{j,j'} = (-1)^{j}
\begin{vmatrix}
1 & l_{j}^{(j')} & l_4^{(j')} & \dots & l_n^{(j')} \\
1 & l_{j}^{(3)} & l_4^{(3)} & \dots & l_n^{(3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & l_{j}^{(n)} & l_4^{(n)} & \dots & l_n^{(n)}
\end{vmatrix}
$$

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and $\Delta_{j,j} = 1$ and $\Delta_{j,j'} = 0$. We choose the linear combination $v_j := (d_2 - d_3)(u_j - I) + (d_3 + e_1)u_{j'}$, which yields

$$
M_j = \begin{vmatrix}
1 & (d_2 - d_3)l_{j'}^{(j')} + (d_3 + e_1)l_j^{(j')} & l_4^{(j')} & \dots & l_n^{(j')} \\
1 & (d_2 - d_3)l_{j'}^{(3)} + (d_3 + e_1)l_j^{(3)} & l_4^{(3)} & \dots & l_n^{(3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & (d_2 - d_3)l_{j'}^{(n)} + (d_3 + e_1)l_j^{(n)} & l_4^{(n)} & \dots & l_n^{(n)}
\end{vmatrix}.
$$

By subtracting appropriate multiples of column 1 from the other columns such that the first row becomes $\begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$ and by expanding the determinant according to the first row, we obtain

(28)
$$
M_j = \begin{vmatrix}
\nu_3 & (l_4^{(3)} - l_4^{(j')}) & \dots & (l_n^{(3)} - l_n^{(j')}) \\
\vdots & \vdots & \ddots & \vdots \\
\nu_n & (l_4^{(n)} - l_4^{(j')}) & \dots & (l_n^{(n)} - l_n^{(j')})
\end{vmatrix},
$$

where $\nu_i := (d_2 - d_3)(l_{j'}^{(i)} - l_{j'}^{(j')}) + (d_3 + e_1)(l_j^{(i)} - l_j^{(j')})$, $i = 3, \dots, n$.

By Lemma 8 we see that

$$
\nu_3 = (d_2 - d_3)(l_{j'}^{(3)} - l_{j'}^{(j')}) + (d_3 + e_1)(l_j^{(3)} - l_j^{(j')}) =
$$
$$
(d_2 - d_3)(d_3 + e_1)\log a + (d_3 + e_1)(d_3 - d_2)\log a + O\left(a^{-1}\right) = O\left(a^{-1}\right)
$$

and

$$
l_k^{(3)} - l_k^{(j')} = d_k \log a - d_k \log a + O\left(a^{-1}\right) = O\left(a^{-1}\right), \qquad 4 \le k \le n,
$$

which proves (23).

In order to prove (24) we note that it is sufficient to show that the coefficient C_j of $\log^{n-3} a$ in the expansion of M_j according to Lemma 8 satisfies $C_j = \Omega\left(a^{-e_1 - d_2 + 1}\right)$.

Lemma 11. *Assume $C_j = O\left(a^{-e_1 - d_2}\right)$. Then for $3 \le k \le n$ we have*
(29a)
$$
C_j = \prod_{l=4}^{k}(-e_l - d_l) \begin{vmatrix}
g_{k,k,j} & g_{k+1,k,j} & g_{k+2,k,j} & \dots & g_{n,k,j} \\
\mu_{k+1,k} & -e_{k+1} - d_{k+1} & 0 & \dots & 0 \\
\mu_{k+2,k} & d_{k+2} - d_{k+1} & -e_{k+2} - d_{k+2} & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mu_{n,k} & d_n - d_{k+1} & d_n - d_{k+2} & \dots & -e_n - d_n
\end{vmatrix},
$$

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where

(29b) $$g_{h,k,j} = l_h^{(3)} - l_h^{(j')}, \qquad k+1 \le h \le n,$$

(29c) $g_{k,k,j}$ *is a $\mathbb{Z}$-linear combination of* $l_i^{(m)}$, $m \in \{j', 3\}$, $i \ne m$,

(29d) $$g_{k,k,j} = O\left(1/a\right),$$

(29e) $$\mu_{h,k} \in \mathbb{N}, \qquad k+1 \le h \le n.$$

Proof. We observe that (29c) and Lemma 8 imply that there are nonnegative integers $\lambda_{i,m,k,j}^+$ and $\lambda_{i,m,k,j}^-$ such that

$$g_{k,k,j} = \sum_{(m,i)} (\lambda_{i,m,k,j}^+ - \lambda_{i,m,k,j}^-) l_i^{(m)} = \tilde{g}_{k,k,j} + O\left(\frac{1}{a^{e_1+d_2}}\right),$$

where

$$\tilde{g}_{k,k,j} = \log \left| 1 + \frac{\prod_{(m,i)}(p_i - p_m)^{\lambda_{i,m,k,j}^+} - \prod_{(m,i)}(p_i - p_m)^{\lambda_{i,m,k,j}^-}}{\prod_{(m,i)}(p_i - p_m)^{\lambda_{i,m,k,j}^-}} \right|$$

and where the indices (i,m) range over $m \in \{j', 3\}, i \ne m$. (29d) asserts that the degree of the numerator is less than the degree of the denominator, hence there is a nonzero integer $\chi_{k,j}$ and a $s_{k,j} \in \mathbb{N}$ such that

(30) $$g_{k,k,j} = \frac{\chi_{k,j}}{a^{s_{k,j}}}\left(1 + O\left(\frac{1}{a}\right)\right) + O\left(\frac{1}{a^{e_1+d_2}}\right).$$

We will now prove the lemma by induction on k.

Let $k = 3$. The definition of C_j, Lemma 8 and the fact that the first row in the determinant in (28) is $O\left(a^{-1}\right)$ necessitate a term containing $\log a$ in rows 2 to $n-2$ in order to get $\log^{n-3} a$. Since $l_h^{(k)} - l_h^{(j')} = O\left(a^{-1}\right)$ for $h > k$, we obtain the representation (29a). (29b) is clear and

(31) $$g_{3,3,j} = (d_2 - d_3)\left(l_{j'}^{(3)} + \sum_{h \ne j'} l_h^{(j')}\right) + (d_3 + e_1)(l_j^{(3)} - l_j^{(j')})$$

implies (29c). (29d) has already been observed and (29e) follows from (4) since for $4 \le h \le n$,

(32) $\mu_{h,3} = (d_2 - d_3)(d_h + e_1) + (d_3 + e_1)(d_h - d_2) = (d_2 + e_1)(d_h - d_3) \in \mathbb{N}$.

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Assume now that the lemma holds for some $3 \leq k \leq n - 1$. Then by (29a), (30) and Lemma 8 we have with $s'_{k,j} := \min\{d_{k+1} - d_3, s_{k,j}\} + 1$

$$
C_j = \prod_{h=4}^{k}(-e_h - d_h)
\begin{vmatrix}
\frac{\chi_{k,j}}{a^{s_{k,j}}} & -\frac{1}{a^{d_{k+1}-d_3}} & 0 & \cdots & 0 \\
\mu_{k+1,k} & -e_{k+1} - d_{k+1} & 0 & \cdots & 0 \\
\mu_{k+2,k} & d_{k+2} - d_{k+1} & -e_{k+2} - d_{k+2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mu_{n,k} & d_n - d_{k+1} & d_n - d_{k+2} & \cdots & -e_n - d_n,
\end{vmatrix}
$$

$$
+ O\left(\frac{1}{a^{s'_{k,j}}}\right)
$$

$$
= \prod_{h=4}^{k}(-e_h - d_h)\left(\frac{\chi_{k,j}}{a^{s_{k,j}}}(-e_{k+1} - d_{k+1}) + \frac{1}{a^{d_{k+1}-d_3}}\mu_{k+1,k}\right) \times
$$

$$
\prod_{h=k+2}^{n}(-e_h - d_h) + O\left(\frac{1}{a^{s'_{k,j}}}\right).
$$

By (29e), (4) and $\chi_{k,j} \neq 0$, we see that if $s_{k,j} \neq d_{k+1} - d_3$, then (24) follows immediately. So we only have to consider the case

$$
(33) \qquad s_{k,j} = d_{k+1} - d_3 =: s_k, \qquad \chi_{k,j} = \frac{\mu_{k+1,k}}{e_{k+1} + d_{k+1}} =: \chi_k \in \mathbb{N}.
$$

(29a) yields

$$
C_j = \prod_{h=4}^{k}(-e_h - d_h)\times
$$

$$
\begin{vmatrix}
g_{k,k,j} + \chi_k g_{k+1,k,j} & g_{k+1,k,j} & g_{k+2,k,j} & \cdots & g_{n,k,j} \\
\mu_{k+1,k} - \chi_k(e_{k+1} + d_{k+1}) & -e_{k+1} - d_{k+1} & 0 & \cdots & 0 \\
\mu_{k+2,k} + \chi_k(d_{k+2} - d_{k+1}) & d_{k+2} - d_{k+1} & -e_{k+2} - d_{k+2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mu_{n,k} + \chi_k(d_n - d_{k+1}) & d_n - d_{k+1} & d_n - d_{k+2} & \cdots & -e_n - d_n
\end{vmatrix}
$$

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and by (33) we obtain

$$C_j = \prod_{h=4}^{k}(-e_h - d_h) \times$$

$$\begin{vmatrix} g_{k+1,k+1,j} & g_{k+1,k,j} & g_{k+2,k+1,j} & \cdots & g_{n,k+1,j} \\ 0 & -e_{k+1} - d_{k+1} & 0 & \cdots & 0 \\ \mu_{k+2,k+1} & d_{k+2} - d_{k+1} & -e_{k+2} - d_{k+2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n,k+1} & d_n - d_{k+1} & d_n - d_{k+2} & \cdots & -e_n - d_n \end{vmatrix},$$

where

(34a) $\qquad g_{h,k+1,j} := g_{h,k,j} \qquad\qquad\qquad\qquad k+2 \le h \le n,$

(34b) $\qquad g_{k+1,k+1,j} := g_{k,k,j} + \chi_k g_{k+1,k,j},$

(34c) $\qquad \mu_{h,k+1} := \mu_{h,k} + \chi_k(d_h - d_{k+1}) \qquad k+2 \le h \le n,$

hence we have

$$C_j = \prod_{h=4}^{k+1}(-e_h - d_h) \begin{vmatrix} g_{k+1,k+1,j} & g_{k+2,k+1,j} & \cdots & g_{n,k+1,j} \\ \mu_{k+2,k+1} & -e_{k+2} - d_{k+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n,k+1} & d_n - d_{k+2} & \cdots & -e_n - d_n \end{vmatrix},$$

which implies (29a) for $k+1$. (34a) yields (29b), (34b) and (29b) result in (29c) and (29d), and (34c), (33), and (4) give (29e) for $k+1$. $\qquad\square$

By (34c) and (33) we see that the recursive definition of ψ_i in (3) matches the definition of χ_i, so that $\chi_i = \psi_i$ for $3 \le i \le n-1$.

(34b) and (31) yield

$$g_{n,n,j} = \left[(d_3 + e_1)l_j^{(3)} + \sum_{k=4}^{n} \psi_{k-1}l_k^{(3)} \right]$$

$$- \left[(e_1 + 2d_3 - d_2)l_j^{(j')} + (d_3 - d_2)l_3^{(j')} + (d_3 - d_2)l_{j'}^{(3)} \right.$$

$$\left. + \sum_{k=4}^{n}(\psi_{k-1} + d_3 - d_2)l_k^{(j')} \right].$$

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From Lemma 8 follows

$$g_{n,n,j} + O\left(\frac{1}{a^{e_1+d_2}}\right) = \log\left(\frac{Q_j^+}{Q_j^-}\right) = \log\left(1 + \frac{Q_j^+ - Q_j^-}{Q_j^-}\right),$$

where Q_j^+ and Q_j^- have been defined in Theorems 1 and 2. Assume that (24) does not hold. This implies $C_j = O\left(a^{-e_1-d_2}\right)$, and so we can apply Lemma 11 to deduce that $\psi_i \in \mathbb{N}$, $3 \le i \le n-1$, and $g_{n,n,j} = O\left(a^{-e_1-d_2}\right)$. But in this case, (5) can be used to show $g_{n,n,j} = \Omega\left(a^{-e_1-d_2+1}\right)$, which yields a contradiction. Hence (24) and therefore Proposition 10 are proved.

8   Large Solutions

To exclude "large" solutions linear forms in logarithms can be employed in the usual procedure. Since we do not care about constants, we will use the explicit bound for the solutions of Thue equations due to Bugeaud and Győry [5].

Theorem 12 (Bugeaud-Győry [5]). *Let $F \in \mathbb{Z}[X,Y]$ be a homogeneous irreducible polynomial of degree $n \ge 3$ and $0 \ne m \in \mathbb{Z}$. Let $B \ge \max\{|m|, e\}$, α be a zero of $F(X,1)$, $K := \mathbb{Q}(\alpha)$, $R := R_K$ the regulator and r the unit rank of K. Let $H \ge 3$ be an upper bound for the absolute values of the coefficients of F.*
Then all solutions $(x,y) \in \mathbb{Z}^2$ of

$$F(x,y) = m$$

satisfy

$$\max\{|x|, |y|\} < \exp\left(C_1 \cdot R \cdot \max\{\log R, 1\} \cdot (R + \log(HB))\right),$$

where

$$C_1 = C_1(n,r) = 3^{r+27}(r+1)^{7r+19}n^{2n+6r+14}.$$

In our situation, we have $m = 1$, $B = e$, $R_K \le R_{\mathfrak{O}} \le R_G = O\left(\log^{n-1} a\right)$ by (17), $r = n-1$, $H = a^{O(1)}$, which yields

$$\log|y| = O\left(\log^{2n-1} a\right).$$

This is a contradiction to Proposition 10, hence there are no solutions with $|y| \ge 2$ for sufficiently large values of a, which proves Theorems 1 and 2.

9 Proof of Corollary 3

If $\delta_4 = 1$, we have $0 < \psi_3 = (e_2 + d_2)/(e_4 + d_4) < 1$ so that $\psi_3 \notin \mathbb{N}$ and we are done. Therefore, we can assume $d_4 - d_3 > 1$.

Calculating Q_j^+ and Q_j^-, we obtain

$$Q_j^+ = a^{\deg Q_j^+} \left(1 + \left((e_1 + d_3)(c_3 - \delta_{j2}\delta_3) + \sum_{k=4}^{n} \psi_{k-1}c_k \right) \frac{1}{a} + O\left(\frac{1}{a^2}\right) \right)$$

$$Q_j^- = a^{\deg Q_j^-} \left(1 + \left((e_1 - d_2 + 2d_3)(c_2 - \delta_2) + 2(d_3 - d_2)(c_3 - \delta_{j1}\delta_3) \right. \right.$$

$$\left. \left. + \sum_{k=4}^{n} (\psi_{k-1} + d_3 - d_2)c_k \right) \frac{1}{a} + O\left(\frac{1}{a^2}\right) \right),$$

where $\deg Q_j^+ = \deg Q_j^- = d_3(e_1 + d_3) + \sum_{k=4}^{n} d_k \psi_{k-1}$, δ_i has been defined in Corollary 3 and δ_{kl} denotes the usual Kronecker-Delta, $\delta_{kl} = 1$ if $k = l$ and 0 otherwise.

We easily see that (11) implies

$$\deg(Q_j^+ - D_j^-) = \deg Q_j^- - 1 > \deg Q_j^- - e_1 - d_2,$$

hence Theorem 1 (or Theorem 2) can be applied.

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