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ON EXPLICIT BOUNDS FOR THE SOLUTIONS OF A CLASS OF PARAMETRIZED THUE EQUATIONS OF ARBITRARY DEGREE

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Abstract. In a recent paper [7] the author considered the family of parametrized Thue equations

$$F_a(X,Y) := \prod_{i=1}^{n}(X - p_i(a)Y) - Y^n = \pm 1, \qquad a \in \mathbb{N}$$

for monic polynomials $p_1, \ldots, p_n \in \mathbb{Z}[a]$ which satisfy

$$\deg p_1 < \cdots < \deg p_n.$$

Under some technical hypothesis it could be proved that there is a computable constant $a_0 = a_0(p_1, \ldots, p_n)$ such that for all integers $a \geq a_0$ the only integer solutions (x, y) of the Diophantine equation satisfy $|y| \leq 1$.

In this paper, we give an explicit expression for a_0 depending on the polynomials $p_1, \ldots, p_n$.

1. Introduction

A *Thue equation* is a Diophantine equation

$$F(X,Y) = m,$$

where $F \in \mathbb{Z}[X,Y]$ is an irreducible form of degree at least 3 and m is a nonzero integer. A. Thue [18] proved in 1909 that the number of integer solutions is finite. A. Baker [1] could give an effective upper bound for the solutions. Recent explicit upper bounds are due to Bugeaud and Győry [3]. Algorithms for the solution of a single Thue equation have been developed by Pethő and Schulenberg [11], Tzanakis and de Weger [19], and Bilu and Hanrot [2].

Starting with E. Thomas [16], parametrized families of Thue equations have been considered (see [9] for further references). In all these cases, an explicit constant a_0 could be given such that there are only "trivial" solutions if the parameter is larger than a_0.

A further step is the investigation of classes of parametrized families of arbitrary degree such as

$$F_a(X,Y) := \prod_{i=1}^{n}(X - p_i(a)Y) - Y^n = \pm 1 \tag{1}$$

where $p_1, \ldots, p_n \in \mathbb{Z}[a]$ are some polynomials, cf. [6, 8, 10, 7]. In these papers, the existence of a constant a_0 in the above sense could be proved. To obtain the constant for a specific family determined by some specific polynomials $p_1, \ldots, p_n$ it is necessary to run along the lines of the proofs and to make all implicit constants in asymptotic arguments explicit.

It is the aim of the present paper to give an explicit expression for a_0 in the case of monic polynomials p_i with increasing degrees. Thomas [17] conjectured the existence of such a constant, this conjecture could be proved under certain technical assumptions in [7].

2. Main Results

To refer to results in [7] we will use the convention that item I.n means item n in [7].

The following notations and assumptions will be used throughout the paper.

Let $n \in \mathbb{N}$, $n \geq 3$ and $p_i \in \mathbb{Z}[a]$ be monic polynomials for $i = 1, \ldots, n$.

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Let $d_i := \deg p_i$ (using the convention $\deg 0 := -1$), $i = 1, \dots, n$, and let the absolute values of all coefficients of $p_1, \dots, p_n$ be bounded by P. Furthermore, we assume

$$d_1 < d_2 < \cdots < d_{n-1} < d_n. \tag{2}$$

Let

$$a_0 := \exp\left(1.01(n+1)(n-1)!(n-1)^{n-2}\exp\bigl(1.04(n-2)(nd_n - n + 3)\bigr)\binom{nd_n - 1}{n-3}(2P+1)^{nd_n}\right). \tag{3}$$

With these notations, we can prove the following explicit version of Theorem I.1:

Theorem 1. *Let $n \geq 4$. Define*

$$e_i := (i-1)d_i + \sum_{l=i+1}^{n} d_l, \qquad 1 \leq i \leq n, \tag{4}$$

and

$$\psi_i := \frac{(e_2 + d_2)(d_{i+1} - d_3)}{e_{i+1} + d_{i+1}} + \sum_{h=3}^{i-1} \frac{d_{i+1} - d_{h+1}}{e_{i+1} + d_{i+1}} \psi_h, \qquad 3 \leq i \leq n-1. \tag{5}$$

If $\psi_i \in \mathbb{N}$ for all $3 \leq i \leq n-1$, we define for $(j, j') \in \{(1,2),(2,1)\}$

$$Q_j^+ := (p_3 - p_j)^{e_1 + d_3} \prod_{k=4}^{n} (p_k - p_3)^{\psi_{k-1}},$$

$$Q_j^- := (p_2 - p_1)^{e_1 + 2d_3 - d_2} (p_3 - p_{j'})^{2(d_3 - d_2)} \prod_{k=4}^{n} (p_k - p_{j'})^{\psi_{k-1} + d_3 - d_2}.$$

If there is a $3 \leq k \leq n-1$ such that $\psi_k \notin \mathbb{N}$ or if we have

$$\deg(Q_j^+ - Q_j^-) > \deg Q_j^- - e_1 - d_2 \tag{6}$$

for $(j, j') = (1,2)$ and for $(j, j') = (2,1)$, then the Diophantine equation (1) only has the solutions

$$(\pm 1, 0) \text{ and } \pm(p_i(a), 1), 1 \leq i \leq n \tag{7}$$

for all integers $a \geq a_0$.

As in [7], the case $n = 3$ has been excluded in the formulation of Theorem 1 in order to avoid any ambiguities; it is stated explicitely in the following theorem as an explicit version of Theorem I.2:

Theorem 2. *Let $n = 3$ and $d_2 \geq 1$. Define $e_1 := d_2 + d_3$. For $(j, j') \in \{(1,2),(2,1)\}$ we define*

$$Q_j^+ := (p_3 - p_j)^{e_1 + d_3},$$
$$Q_j^- := (p_2 - p_1)^{e_1 + 2d_3 - d_2}(p_3 - p_{j'})^{2(d_3 - d_2)}.$$

If we have

$$\deg(Q_j^+ - Q_j^-) > \deg Q_j^- - e_1 - d_2$$

for $(j, j') = (1,2)$ and for $(j, j') = (2,1)$, then the Diophantine equation (1) only has the solutions

$$(\pm 1, 0) \text{ and } \pm(p_i(a), 1), 1 \leq i \leq 3$$

for all integers $a \geq a_0$.

Weaker formulations of the technical hypothesis corresponding to Corollary I.3 can be given as follows:

Corollary 3. *We write*

$$p_i(a) = a^{d_i} + c_i a^{d_i - 1} + \text{ terms of lower degree}, \qquad i = 2, \dots, n.$$

Let

$$\delta_i := \begin{cases} 1 & \text{if } d_i - d_{i-1} = 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$e := \sum_{i=2}^{n} d_i.$$

If $\delta_4 = 1$ *or*

$$(e - d_2 + 2d_3)(c_2 - \delta_2) + (-e - 2d_2 + d_3)c_3 + (d_3 - d_2)\sum_{i=4}^{n} c_i \notin \{2\delta_3, -(e + d_3)\delta_3\}, \tag{8}$$

then the Diophantine equation (1) *only has the solutions* (7) *for all integers* $a \geq a_0$.

In particular, this implies that if $\deg p_4 = \deg p_3 + 1$, then (1) only has the solutions (7) for $a \geq a_0$.

The proof of Corollary 3 is identical to the proof of Corollary I.3.

3. Preliminaries

While our final result can only be proved for $a \geq a_0$, intermediate results will hold for smaller values of a. We collect the corresponding bounds here:

$$\begin{aligned} a_1 &:= 2P + 2, \\ a_2 &:= 1.8n^2(1.1)^n P \geq 21P, \\ a_3 &:= 3n^n P, \\ a_4 &:= \exp\bigl(38n^n d_n^n 2^{nd_n} P^{nd_n}\bigr). \end{aligned}$$

Note that $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_0$ for $n \geq 3$, $P \geq 1$ and $d_n \geq \min(2, n-2)$ (this last relation is a consequence of (2)). a will always denote a positive integer.

As an analogue to the usual O-Notation we use an "L-Notation" (borrowed from de Bruijn [4, Section 1.2]): $f(a) = L(g(a))$ will mean $|f(a)| \leq g(a)$, and we will use it in the middle of a formula in the same way as the O-Notation. For brevity, we will write p_i instead of $p_i(a)$ in many situations.

Lemma 4. *If* $a \geq a_1$ *and* $1 \leq i \neq j \leq n$, *then*

$$|p_i(a) - p_j(a)| \geq \frac{a^{\max(d_i, d_j)}}{2P + 1},$$

which implies

$$|p_i(a) - p_j(a)| \geq 1.$$

Proof. Without loss of generality we may assume $i > j$. By (2) we get

$$|p_i(a) - p_j(a)| \geq a^{d_i} - 2P\sum_{k=0}^{d_i - 1} a^k \geq a^{d_i}\left(1 - \frac{2P}{a - 1}\right)$$

which proves the first assertion of the lemma. The second follows since both $p_i(a)$ and $p_j(a)$ are integers. □

Checking the proof of Lemma I.5 using Lemma 4, we obtain

Lemma 5. *For* $a \geq a_1$, *the solutions* (x, y) *of* (1) *with* $|y| \leq 1$ *are precisely those listed in* (7).

We consider the polynomial $f_a(X) := F_a(X, 1)$ and give asymptotic estimates for its roots $\alpha^{(1)}, \ldots, \alpha^{(n)}$ similar to Lemma I.6:

Lemma 6. *Let* $a \geq a_2$. *All roots of* f_a *are real and fulfill the estimates*

$$\alpha^{(i)} = p_i + \frac{(-1)^{n-i}}{a^{e_i}} + L\left(\frac{2.1nP}{a^{e_i+1}}\right) = p_i + L\left(\frac{1.3}{a^{e_i}}\right) = p_i + L\left(\frac{1.3}{a^3}\right), \qquad i = 1, \ldots, n. \tag{9}$$

Proof. Fix some $1 \le i \le n$. For $k = 1, 2$ we define

$$\alpha_{i,k} := p_i + \frac{(-1)^{n-i}}{a^{e_i}} \left(1 + (-1)^k \frac{2.1nP}{a}\right)$$

and obtain for $j \ne i$

$$\alpha_{i,k} - p_j = a^{\deg(p_i - p_j)} (-1)^{\sigma_{ij}} \left(1 + L\left(\frac{2P}{a-1}\right)\right),$$

where $\sigma_{ij} = 0$ for $i > j$ and $\sigma_{ij} = 1$ for $i < j$. This implies

$$f(\alpha_{i,1}) \le \left(1 - \frac{2.1nP}{a}\right)\left(1 + \frac{2.1P}{a}\right)^{n-1} - 1,$$
$$f(\alpha_{i,2}) \ge \left(1 + \frac{2.1nP}{a}\right)\left(1 - \frac{2.1P}{a}\right)^{n-1} - 1$$

by (I.14). For $0 \le z \le 7/(6n^2(1.1)^n)$ Taylor's formula yields

$$(1 - nz)(1 + z)^{n-1} < 1,$$
$$(1 + nz)(1 - z)^{n-1} > 1.$$

Therefore we get $f(\alpha_{i,1}) < 0 < f(\alpha_{i,2})$ which proves that there is a real zero $\alpha^{(i)}$ satisfying (9). Lemma 4 shows that all roots $\alpha^{(i)}$ which we find using this method are distinct. □

4. Associated Number Field

By Lemma 4 and [8, Proposition 3], f_a is an irreducible polynomial for $a \ge a_1$. Therefore the number field $K := \mathbb{Q}(\alpha)$ generated by one of the roots $\alpha = \alpha^{(i)}$ of f_a has degree n over $\mathbb{Q}$. From Section I.4 we recall that this implies that solutions $(x, y) \in \mathbb{Z}^2$ of (1) correspond to units $x - \alpha y$ in $\mathfrak{O} := \mathbb{Z}[\alpha]$. As in [7], we define units $\eta_i := \alpha - p_i$ and the abbreviation $l_i^{(k)} := \log\left|\eta_i^{(k)}\right|$ with $\eta_i^{(k)} = \alpha^{(k)} - p_i$.

We will need explicit estimates for the $l_i^{(k)}$ as in Lemma I.8:

Lemma 7. *Let $1 \le i, k \le n$, $m := \min(i, k)$, $M := \max(i, k)$ and $a \ge a_2$. Then we have*

$$l_i^{(k)} = \log(p_M - p_m) + L\left(\frac{2.6(2P+1)}{a^{e_m + d_M}}\right), \qquad i \ne k, \tag{10}$$

and in particular

$$l_i^{(k)} = \begin{cases} d_M \log a + L\left(\frac{4.2P}{a}\right) & \text{if } i \ne k, \\ -e_i \log a + L\left(\frac{4.2(n-1)P}{a}\right) & \text{if } i = k. \end{cases} \tag{11}$$

If $1 \le L \le e_m + d_M$ then there are $r_{M,m,l} \in \mathbb{Q}$, $0 \le l \le L - 1$, which depend only on the coefficients of the polynomials p_s, $1 \le s \le n$, such that

$$l_i^{(k)} = r_{M,m,0} \log a + \sum_{l=1}^{L-1} \frac{r_{M,m,l}}{a^l} + L\left(\frac{1.5(n-1)(2P+1)^L}{La^L}\right), \tag{12}$$

$$r_{M,m,l} \in \mathbb{Z}\left[\frac{1}{\operatorname{lcm}(1, \dots, l)}\right], \tag{13}$$

$$|r_{M,m,l}| \le \frac{1}{l}(n-1)(2P+1)^l, \qquad 1 \le l \le L - 1. \tag{14}$$

Proof. Assume $i \ne k$. By definition and by (2) we obtain

$$l_i^{(k)} = \log\left|\eta_i^{(k)}\right| = \log\left|p_k - p_i + \alpha^{(k)} - p_k\right| = \log(p_M - p_m) + \log\left|1 + \frac{\alpha^{(k)} - p_k}{p_k - p_i}\right|.$$

By Lemma 6 and Lemma 4 we get for $a \geq a_2$

$$\left|\frac{\alpha^{(k)} - p_k}{p_k - p_i}\right| \leq \frac{1.3(2P+1)}{a^{e_m + d_M}} < \frac{1}{2}.$$

Since for $|z| \leq 1/2$ we have $|\log(1+z)| \leq 2\,|z|$, this proves (10).

The observation

$$\log(p_M - p_m) = \log\left(a^{d_M}\left(1 + L\left(\frac{2P}{a-1}\right)\right)\right)$$

similarly yields (11) for $i \neq k$. The case $i = k$ can be reduced to the case $i \neq k$ because the relation $l_i^{(i)} = -\sum_{j \neq i} l_j^{(i)}$ holds by definition of $\alpha^{(i)}$. This results in the factor $(n-1)$ in (11).

In order to prove the remaining part of the lemma, we introduce integer coefficients $c_{M,m,s}$, $1 \leq s \leq d_M$, for $i \neq k$ such that

$$\begin{aligned}\log(p_M - p_m) &= \log\left(a^{d_M}\left(1 + \sum_{s=1}^{d_M} \frac{c_{M,m,s}}{a^s}\right)\right)\\ &= d_M \log a + \sum_{t=1}^{\infty} \frac{(-1)^{t+1}}{t}\left(\sum_{s=1}^{d_M} \frac{c_{M,m,s}}{a^s}\right)^t.\end{aligned}$$

Defining $r_{M,m,l}$ to be the coefficient of a^{-l} in this expansion, we obtain

$$r_{M,m,l} = \sum_{t=1}^{l} \frac{(-1)^{t+1}}{t} \sum_{\substack{0 \leq s_1, \ldots, s_t \leq d_M - 1\\ s_1 + \cdots + s_t = l - t}} \prod_{\nu=1}^{t} c_{M,m,s_\nu + 1}. \tag{15}$$

This immediately proves (13).

Since by definition $|c_{M,m,s}| \leq 2P$, we can estimate $r_{M,m,l}$ by

$$\begin{aligned}|r_{M,m,l}| &\leq \sum_{t=1}^{l} \frac{1}{t}(2P)^t \sum_{\substack{0 \leq s_1, \ldots, s_t\\ s_1 + \cdots + s_t = l - t}} 1\\ &\leq \sum_{t=1}^{l} \frac{1}{t}(2P)^t \binom{l-1}{t-1}\\ &= \frac{1}{l}(2P+1)^l.\end{aligned}$$

This yields (14) (the factor $(n-1)$ in (14) is needed for $i = k$).

Finally, we have to prove the remainder term in (12):

$$\left|\sum_{l=L}^{\infty} \frac{r_{M,m,l}}{a^l}\right| \leq \frac{(2P+1)^L}{La^L} \sum_{l=0}^{\infty} \left(\frac{2P+1}{a}\right)^l \leq \frac{1.17(2P+1)^L}{La^L}.$$

Taking into account the remainder term from (10) and the case $i = k$, we get (12). □

We will show that η_i, $i = 1, \ldots, n-1$, are "sufficiently close" to fundamental units in $\mathfrak{O}^\times$. To achieve this aim, we will need some lower bound for the regulator R_K of the number field. We take an absolute bound of Pohst [13, Satz II]:

Lemma 8 (Pohst)**.** *Let K be a totally real number field. Then the regulator R_K satisfies*

$$R_K > 0.315.$$

We remark that we could choose a bound which depends on the discriminant of the number field (cf. Pohst [12]). We would gain a logarithmic factor, but the constants would be harder to deal with (and the final constant a_0 would not be improved).

In order to estimate determinants involving our asymptotic bounds, we need the following auxiliary result:

Lemma 9. *Let C and Δ be $n \times n$ matrices with columns $c_1, \dots, c_n$ and $\delta_1, \dots, \delta_n$ respectively. Let $\|c_i\|_2 < \varrho$ and $\|\delta_i\|_2 < \varepsilon\varrho$ for $1 \le i \le n$.*

If

$$\varepsilon < \min\left(0.1, \frac{2}{n(1.1)^n}\right),$$

then we have

$$\det(C + \Delta) = \det C + L\left(2n\varrho^n\varepsilon\right) \tag{16}$$

Proof. We can express the determinant under consideration as

$$\begin{aligned}\det(c_1 + \delta_1, \dots, c_n + \delta_n) &= \det(c_1, \dots, c_n) + \det(c_1, \dots, c_{n-1}, \delta_n) \\ &\quad + \det(c_1, \dots, c_{n-2}, \delta_{n-1}, c_n + \delta_n) + \dots + \det(\delta_1, c_2 + \delta_2, \dots, c_n + \delta_n).\end{aligned}$$

Using Hadamard's inequality, we obtain

$$\begin{aligned}\det(C + \Delta) &= \det C + L\left(\varepsilon\varrho^n\right) + L\left(\varepsilon\varrho^n(1+\varepsilon)\right) + \dots + L\left(\varepsilon\varrho^n(1+\varepsilon)^{n-1}\right) \\ &= \det C + L\left(\varepsilon\varrho^n \frac{(1+\varepsilon)^n - 1}{\varepsilon}\right).\end{aligned}$$

For the given range of ε, this implies (16). □

In our applications of Lemma 9 it will be convenient to refer to the following lemma:

Lemma 10. *Let $a \ge a_2$, $1 \le k \le n$ and $\{i_1, \dots, i_{n-1}\}$ be a subset of $\{1, \dots, n\}$ of cardinality $n-1$. Then*

$$\left\| \left(l_i^{(k)}\right)_{i=i_1,\dots,i_{n-1}} \right\|_2 \le n d_n \log a.$$

Proof. This is a consequence of (11) and (2). □

We now have collected all tools to prove the following analogue of Lemma I.9:

Lemma 11. *Let $\{i_1, \dots, i_{n-1}\}$ be a subset of $\{1, \dots, n\}$ of cardinality $n-1$ and*

$$G := \left\langle -1, \eta_{i_1}, \dots, \eta_{i_{n-1}} \right\rangle \subseteq \mathfrak{O}^\times.$$

Define

$$D := \left| \det \begin{pmatrix} -e_1 & d_2 & d_3 & \dots & d_{n-1} \\ d_2 & -e_2 & d_3 & \dots & d_{n-1} \\ d_3 & d_3 & -e_3 & \dots & d_{n-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ d_{n-1} & d_{n-1} & d_{n-1} & \dots & -e_{n-1} \end{pmatrix} \right|$$

Then the regulator R_G can be estimated by

$$R_G = D \log^{n-1} a + L\left(8.4 n^n d_n^{n-2} P \frac{\log^{n-2} a}{a}\right) \tag{17}$$

if $a \ge a_2$. For $a \ge a_3$ we conclude that

$$\frac{1}{2} D \log^{n-1} a \le R_G \le \frac{3}{2} D \log^{n-1} a. \tag{18}$$

For $a \ge a_3$ the index $[\mathfrak{O}^\times : G]$ is bounded by

$$[\mathfrak{O}^\times : G] \le 4.8 D \log^{n-1} a. \tag{19}$$

Proof. Assume first $i_1 = 1, \dots, i_{n-1} = n-1$. Equation (11), Lemma 9 and Lemma 10 imply

$$R_G = D \log^{n-1} a + L\left(2(n-1)(n d_n \log a)^{n-1} \frac{4.2P}{d_n a \log a}\right),$$

and we obtain (17).

Gershgorin's circle theorem [5] shows that $D \ge d_n^{n-1}$, which leads to (18) for $a \ge a_3$.

For arbitrary $i_1, \dots, i_{n-1}$, the result follows from $l_n^{(i)} = -\sum_{k=1}^{n-1} l_k^{(i)}$.

Equation (19) is a consequence of Pohst and Zassenhaus [14, p. 361], Lemma 8, and (18):

$$I = [\mathfrak{O}^\times : G] = \frac{R_G}{R_{\mathfrak{O}}} \leq \frac{R_G}{R_{\mathfrak{O}_K}} \leq \frac{R_G}{0.315} \leq 4.8 D \log^{n-1} a.$$

□

5. Approximation Properties of Solutions

For a solution $(x, y) \in \mathbb{Z}^2$ of (1) we define $\beta := x - \alpha y$. We say that (x, y) is a solution of *type* j if

$$\left|\beta^{(j)}\right| = \min_{i=1,\dots,n} \left|\beta^{(i)}\right|.$$

The standard machinery for Thue equations yields

Lemma 12. *For $a \geq Pa_2$ the estimates*

$$\left|\beta^{(j)}\right| \leq 2^{n-1}(2P+2)^{n-1} \frac{1}{|y|^{n-1}} \cdot \frac{1}{a^{e_j}}, \tag{20}$$

$$\log\left|\beta^{(i)}\right| = \log|y| + l_j^{(i)} + L\left(\frac{(2P+2)^n}{a^{e_j+d_2}}\right), \qquad i \neq j, \tag{21}$$

hold.

Proof. Since $|y|\left|\alpha^{(i)} - \alpha^{(j)}\right| \leq 2\left|\beta^{(i)}\right|$, we obtain

$$\left|\beta^{(j)}\right| = \frac{1}{\prod_{i \neq j}\left|\beta^{(i)}\right|} \leq \frac{2^{n-1}}{|y|^{n-1}\prod_{i\neq j}\left|\alpha^{(i)} - \alpha^{(j)}\right|}.$$

Estimating $\left|\alpha^{(i)} - \alpha^{(j)}\right|$ by (9) and Lemma 4 results in (20).

Since

$$\left|\frac{\beta^{(i)}}{y}\right| = \left|\frac{x}{y} - \alpha^{(j)} + \alpha^{(j)} - p_j + p_j - \alpha^{(i)}\right| = \left|\alpha^{(i)} - p_j\right| \cdot \left|1 + \frac{\alpha^{(j)} - p_j}{p_j - \alpha^{(i)}} + \frac{\beta^{(j)}}{y(p_j - \alpha^{(i)})}\right|,$$

estimate (21) follows from Lemma 6 and (20). □

The main task is to exclude solutions with $|y| \geq 1$ but $|y|$ not very large. To this aim, we prove the following analogue to Proposition I.10:

Proposition 13. *Let $(x, y) \in \mathbb{Z}^2$ be a solution of* (1) *with $|y| \geq 2$ and $a \geq a_0$. Then*

$$\log|y| \geq \frac{0.05}{1.2^n P n^{n-2} d_n^{2n-5}} \cdot \frac{a}{\log^{n-3} a}. \tag{22}$$

Proof. Since β is a unit by (I.16), Lemma 11 yields

$$\beta^I = \pm \eta_{i_1}^{u_{i_1}} \dots \eta_{i_{n-1}}^{u_{i_{n-1}}}, \tag{23}$$

where $\{i_1, \dots, i_{n-1}\}$ is a subset of $\{1, \dots, n\}$ of cardinality $n-1$, which will be chosen depending on the case j of the solution, $u_{i_1}, \dots, u_{i_{n-1}}$ are integers and I can be bounded by (19).

Taking logarithms of the conjugates $h \in \{1, \dots, n\} \setminus \{j\}$ of (23), we get a system of linear equations for the u_{i_k}/I:

$$\log\left|\beta^{(h)}\right| = \frac{u_{i_1}}{I} l_{i_1}^{(h)} + \dots + \frac{u_{i_{n-1}}}{I} l_{i_{n-1}}^{(h)}, \qquad h \neq j.$$

Cramer's rule yields

$$R\frac{u_{i_k}}{I} = \begin{vmatrix} l_{i_1}^{(1)} & \dots & l_{i_{k-1}}^{(1)} & \log\left|\beta^{(1)}\right| & l_{i_{k+1}}^{(1)} & \dots & l_{i_{n-1}}^{(1)} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{i_1}^{(n)} & \dots & l_{i_{k-1}}^{(n)} & \log\left|\beta^{(n)}\right| & l_{i_{k+1}}^{(n)} & \dots & l_{i_{n-1}}^{(n)} \end{vmatrix},$$

where the j-th row is omitted and R denotes the determinant of the system matrix, which is (up to a sign) the regulator R_G estimated in Lemma 11.

Applying (21) and Lemma 10 we obtain for $a \geq Pa_2$

$$R\frac{u_{i_k}}{I} = M_{j,i_k}\log|y| + \Delta_{j,k}R + L\left((2P+2)^n d_n^{n-2} n^{n-3/2}\frac{\log^{n-2} a}{a^{e_j+d_2}}\right), \tag{24}$$

where $\Delta_{j,k} = \pm 1$ if $j \notin \{i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_{n-1}\}$ and 0 else and

$$M_{j,i_k} = \begin{vmatrix} l_{i_1}^{(1)} & \dots & l_{i_{k-1}}^{(1)} & 1 & l_{i_{k+1}}^{(1)} & \dots & l_{i_{n-1}}^{(1)} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{i_1}^{(n)} & \dots & l_{i_{k-1}}^{(n)} & 1 & l_{i_{k+1}}^{(n)} & \dots & l_{i_{n-1}}^{(n)} \end{vmatrix},$$

where the j-th row is omitted.

Using the information on $l_i^{(k)}$ contained in Lemma 7, we obtain the following lemma which will be proved at the end of this section.

Lemma 14. *If $a \geq Pa_2$ and $2 \leq L \leq e_1 + d_2$, then there are $G_{j,i,l,\lambda} \in \mathbb{Q}$ for $0 \leq l \leq L-1$ and $0 \leq \lambda \leq n-2$ such that*

$$M_{j,i} = \sum_{l=0}^{L-1} \sum_{\lambda=\max(0,n-2-l)}^{n-2} G_{j,i,l,\lambda}\frac{\log^\lambda a}{a^l} \tag{25}$$

$$+ L\left(0.24 n^{n-2} d_n^{n-3}(2P+1)^L \left(\frac{7}{6}\right)^n L\binom{n+L-3}{n-3}(n-1)!\frac{\log^{n-3} a}{a^L}\right)$$

$$G_{j,i,l,n-2} = 0 \qquad \text{if } l \geq 1 \tag{26}$$

$$|G_{j,i,l,\lambda}| \leq (n-1)!(n-1)^{n-2} d_n^\lambda (2P+1)^l \binom{n-2}{\lambda}\binom{l-1}{n-\lambda-3} \tag{27}$$

and if $G_{j,i,l,\lambda} \neq 0$ then

$$|G_{j,i,l,\lambda}| \geq \exp\bigl(-1.04 \cdot (n-2-\lambda)(\lambda+l-n+3)\bigr). \tag{28}$$

If $j \in \{1,2\}$, we set

$$j' := \begin{cases} 2 & \text{if } j = 1, \\ 1 & \text{if } j = 2 \end{cases}$$

and $(i_1, \dots, i_{n-1}) = (1, 2, 4, \dots, n)$. We choose $v_j := (d_2 - d_3)(u_j - I) + (d_3 + e_1)u_{j'}$ and by (24) we get

$$R\frac{v_j}{I} = M_j \log|y| + L\left((n+1)d_n(2P+2)^n d_n^{n-2} n^{n-3/2}\frac{\log^{n-2} a}{a^{e_j+d_2}}\right), \tag{29}$$

where

$$M_j = \begin{vmatrix} (d_2-d_3)(l_{j'}^{(3)} - l_{j'}^{(j')}) + (d_3+e_1)(l_j^{(3)} - l_j^{(j')}) & (l_4^{(3)} - l_4^{(j')}) & \dots & (l_n^{(3)} - l_n^{(j')}) \\ \vdots & \vdots & \ddots & \vdots \\ (d_2-d_3)(l_{j'}^{(n)} - l_{j'}^{(j')}) + (d_3+e_1)(l_j^{(n)} - l_j^{(j')}) & (l_4^{(n)} - l_4^{(j')}) & \dots & (l_n^{(n)} - l_n^{(j')}) \end{vmatrix}$$

by (I.28).

Equation (11) yields

$$(d_2-d_3)(l_{j'}^{(3)} - l_{j'}^{(j')}) + (d_3+e_1)(l_j^{(3)} - l_j^{(j')}) = L\left(\frac{12.6 \cdot Pnd_n}{a}\right)$$

$$l_k^{(3)} - l_k^{(j')} = L\left(\frac{8.4 \cdot P}{a}\right), \qquad 4 \leq k \leq n.$$

By Hadamard's inequality we obtain

$$M_j \leq 7.5(1.2)^n P n^{n-2} d_n^{2n-5}\frac{\log^{n-3} a}{a}. \tag{30}$$

If $3 \leq j \leq n$, we choose $(i_1, \dots, i_{n-1}) = (1, 3, \dots, n)$ and $v_j := u_1$. By (24), we get (29) for this case, too, where by (I.25)

$$M_j = \begin{vmatrix} 0 & l_3^{(1)} - l_3^{(2)} & \dots & l_n^{(1)} - l_n^{(2)} \\ 1 & l_3^{(2)} & \dots & l_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & l_3^{(n)} & \dots & l_n^{(n)} \end{vmatrix},$$

where the j-th row is omitted. Since by (11) we have

$$l_k^{(1)} - l_k^{(2)} = L\left(\frac{8.4 \cdot P}{a}\right), \qquad 3 \leq k \leq n,$$

(30) holds in this case also.

For all j there is an expansion

$$M_j = \sum_{l=0}^{e_1+d_2-1} \sum_{\lambda=\max(0,n-2-l)}^{n-2} G_{j,l,\lambda} \frac{\log^\lambda a}{a^l} + L\left(0.32 n^n d_n^{n-1} (2P+1)^{e_1+d_2} \left(\frac{7}{6}\right)^n \binom{n(d_n+1)-3}{n-3} (n-1)! \frac{\log^{n-3} a}{a^{e_1+d_2}}\right)$$

for some rationals $G_{j,l,\lambda}$ independent of a by Lemma 14.

In [7] we proved (I.24) assuming the technical hypothesis of Theorems 1 and 2. Therefore, there are some $1 \leq l \leq e_1 + d_2 - 1$ and some $0 \leq \lambda \leq n-3$ — we remark that $\lambda = n-2$ would imply $l = 0$ by (26), which is impossible by (30) — such that $G_{j,l,\lambda} \neq 0$. We choose (l_0, λ_0) such that $G_{j,l_0,\lambda_0} \neq 0$, but $G_{j,l,\lambda} = 0$ for all $(-l, \lambda) \geq_{lex} (-l_0, \lambda_0)$.

By (29), (28), (27) and (25) (for $L = l_0 + 1$), $\log |y| \geq \log 2$, we obtain

$$\begin{aligned} \left|R\frac{v_j}{I}\right| \geq \frac{\log^{\lambda_0} a}{a^{l_0}} \log 2 \Bigg(& \exp\big(-1.04(n-2-\lambda_0)(\lambda_0 + l_0 - n + 3)\big) \\ & - \sum_{\lambda=0}^{\lambda_0 - 1} (n+1) d_n (n-1)! (n-1)^{n-2} d_n^\lambda (2P+1)^{l_0} \binom{n-2}{\lambda} \binom{l_0 - 1}{n-\lambda-3} \log^{\lambda-\lambda_0} a \\ & - 0.24(n+1) n^{n-1} d_n^{n-1} (2P+1)^{n d_n} \left(\frac{7}{6}\right)^n \binom{n + n d_n - 3}{n-3} (n-1)! \frac{\log^{n-3} a}{a} \\ & - \frac{1}{\log 2} (n+1) d_n (2P+2)^n d_n^{n-2} n^{n-3/2} \frac{\log^{n-2} a}{a} \Bigg). \end{aligned}$$

For $a \geq a_4$, we get $\log^{10\,000n} a \leq a$. The above expression is minimal for maximal l_0 which we estimate by $l_0 \leq n d_n$. We note that

$$\binom{n-2}{\lambda} \binom{n d_n - 1}{n - \lambda - 3} \leq \binom{n d_n - 1}{n-3}.$$

For $a \geq a_0$, this implies $|R v_j / I| > 0$, i. e. $|v_j| > 0$, which yields $|v_j| \geq 1$.

Together with (29) and Lemma 11 this implies

$$|M_j \log |y|| \geq \left|\frac{R}{I}\right| \cdot |v_j| - (n+1) d_n (2P+2)^n d_n^{n-2} n^{n-3/2} \frac{\log^{n-2} a}{a^{e_j + d_2}} \geq 0.314.$$

Using (30) we finally obtain (22). □

Proof of Lemma 14. By definition we get

$$M_{j,i_k} = \sum_{\sigma \in S_{j,i_k}} \operatorname{sgn}(\sigma) \prod_{t \in T_{j,i_k}} l_t^{(\sigma(t))},$$

where

$$T_{j,i_k} := \{i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_{n-1}\}$$
$$S_{j,i_k} := \big\{\sigma : T_{j,i_k} \cup \{i_k\} \to \{1, \dots, j-1, j+1, \dots, n\} \text{ bijection}\big\}.$$

From (12) and (14), we obtain

$$\prod_{t\in T_{j,i_k}} l_t^{(\sigma(t))} = \prod_{t\in T_{j,i_k}} \left(r_{M,m,0}\log a + \sum_{l=1}^{L-1} \frac{r_{M,m,l}}{a^l} \right) + L\left(\frac{1.5n^{n-2}d_n^{n-3}(2P+1)^L}{L} \cdot \frac{\log^{n-3} a}{a^L} \right), \tag{31}$$

where M and m are shortcuts for $\max(t, \sigma(t))$ and $\min(t, \sigma(t))$, respectively.

Expanding the product in (31) results in

$$\prod_{t\in T_{j,i_k}} \left(r_{M,m,0}\log a + \sum_{l=1}^{L-1} \frac{r_{M,m,l}}{a^l} \right) = \sum_{l=0}^{(n-2)(L-1)} \frac{1}{a^l} \left(\sum_{\lambda=0}^{n-2} \log^\lambda a \tilde{G}_{j,i,l,\lambda,\sigma} \right), \tag{32}$$

where $\tilde{G}_{j,i,l,\lambda,\sigma} \in \mathbb{Q}$. We remark that if we do not take a term $r_{M,m,0}\log a$ — which occurs $n-2-\lambda$ times — we have to take at least a factor $1/a$, which shows that

$$\tilde{G}_{j,i,l,\lambda,\sigma} = 0 \text{ for } \lambda + l < n-2. \tag{33}$$

Similarly we note that if $\lambda = n-2$ then $l = 0$, which proves (26).

We estimate the denominator of $\tilde{G}_{j,i,l,\lambda,\sigma}$. By (32) and (33), it is the product of $n-2-\lambda$ terms r_{M,m,l_t} with $\sum_t l_t = l$, which implies that for each t we have $l_t \leq l - (n-3-\lambda)$. Therefore (13) yields

$$\operatorname{denominator}(\tilde{G}_{j,i,l,\lambda,\sigma}) \leq \operatorname{lcm}(1, \dots, l + \lambda - (n-3))^{n-2-\lambda}. \tag{34}$$

Rosser and Schoenfeld [15, Theorem 12] prove for $k \in \mathbb{N}$

$$\log \operatorname{lcm}(1, \dots, k) \leq 1.04k.$$

Together with (34) this leads to

$$\operatorname{denominator}(\tilde{G}_{j,i,l,\lambda,\sigma}) \leq \exp(1.04 \cdot (l + \lambda - n + 3)(n-2-\lambda)),$$

and (28) is proved.

Now, we consider upper bounds for $\tilde{G}_{j,i,l,\lambda,\sigma}$: From (32), (11), and (14) we get

$$\begin{aligned} \left|\tilde{G}_{j,i,l,\lambda,\sigma}\right| &\leq \binom{n-2}{\lambda} \big((n-1)d_n\big)^\lambda \sum_{\substack{1\leq l_1,\dots,l_{n-2-\lambda}\\ l_1+\dots+l_{n-2-\lambda}=l}} \prod_{u=1}^{n-2-\lambda} \big((n-1)(2P+1)^{l_u}\big) \\ &\leq (n-1)^{n-2} d_n^\lambda (2P+1)^l \binom{n-2}{\lambda} \binom{l-1}{n-\lambda-3}, \end{aligned} \tag{35}$$

which leads to (27).

We still have to prove the remainder term in (25). Using (26) and (35) we obtain

$$\begin{aligned} &\left| \sum_{l=L}^{(n-2)(L-1)} \frac{1}{a^l} \left(\sum_{\lambda=0}^{n-2} \log^\lambda a \tilde{G}_{j,i,l,\lambda,\sigma} \right) \right| \\ &\leq \frac{\log^{n-3} a}{a^L} (n-1)^{n-2} d_n^{n-3} (2P+1)^L \sum_{l=L}^{\infty} \left(\frac{2P+1}{a} \right)^{l-L} \sum_{\lambda=0}^{n-3} \binom{n-2}{\lambda} \binom{l-1}{n-\lambda-3} \\ &= \frac{\log^{n-3} a}{a^L} (n-1)^{n-2} d_n^{n-3} (2P+1)^L \sum_{l=L}^{\infty} \left(\frac{2P+1}{a} \right)^{l-L} \binom{n+l-3}{n-3}. \end{aligned}$$

To estimate this sum, we note that for $0 \le z < 1$ and integers u, v Cauchy's remainder form for Taylor's theorem used for $F(z) = (1-z)^{-(v+1)}$ yields

$$\sum_{l=0}^{\infty} z^l \binom{u+v+l}{v} \le u \binom{u+v}{u} (1-z)^{-(v+2)}.$$

We conclude that

$$\left| \sum_{l=L}^{(n-2)(L-1)} \frac{1}{a^l} \left(\sum_{\lambda=0}^{n-2} \log^\lambda a \tilde{G}_{j,i,l,\lambda,\sigma} \right) \right| \le \left(\frac{7}{6}\right)^{n-1} (2P+1)^L (n-1)^{n-2} d_n^{n-3} L \cdot \binom{n+L-3}{n-3} \cdot \frac{\log^{n-3} a}{a^L}$$

and combine it with the remainder term from (31) so that we get (25). □

6. Large Solutions

We will now exclude "large solutions" using an explicit bound due to Bugeaud and Győry [3]:

Theorem 15 (Bugeaud-Győry [3])**.** *Let $F \in \mathbb{Z}[X,Y]$ be a homogeneous irreducible polynomial of degree $n \ge 3$ and $0 \ne m \in \mathbb{Z}$. Let $B \ge \max\{|m|, e\}$, α be a zero of $F(X,1)$, $K := \mathbb{Q}(\alpha)$, $R := R_K$ the regulator and r the unit rank of K. Let $H \ge 3$ be an upper bound for the absolute values of the coefficients of F.*

Then all solutions $(x,y) \in \mathbb{Z}^2$ of

$$F(x,y) = m$$

satisfy

$$\max\{|x|, |y|\} < \exp\Big(C \cdot R \cdot \max\{\log R, 1\} \cdot (R + \log(HB))\Big),$$

where

$$C = C(n,r) = 3^{r+27}(r+1)^{7r+19} n^{2n+6r+14}.$$

In our situation, we have $B = e$, $R_K \le R_{\mathfrak{O}} \le R_G \le \frac{3}{2} D \log^{n-1} a$ by (18), $r = n-1$. After some calculations, we get

$$H \le 1.01 a^{n d_n}$$

for $a \ge a_0$. We obtain

$$\log|y| \le 5.78 \cdot 10^{12} \cdot 3^n n^{17n+19} d_n^{2n-2} \log^{2n-2} a \log\log a.$$

By (22), we get

$$a \le 1.16 \cdot 10^{14} \cdot (3.6)^n d_n^{4n-7} n^{18n+17} P \log^{3n-5} a \log\log a.$$

This leads to a contradiction to $a \ge a_0$, which proves Theorems 1 and 2.

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