Another well-solvable case of the QAP: Maximizing the job completion time variance

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Abstract

We analyze a special case of the maximum quadratic assignment problem where one matrix is a monotone anti-Monge matrix and the other matrix has a multi-layered structure that is built on top of certain Toeplitz matrices. As an application of our main result, we derive a (simple and concise) alternative proof for a recent result on the scheduling problem of maximizing the variance of job completion times.

Keywords: Combinatorial optimization; quadratic assignment problem; machine scheduling; well-solvable special case.

1. Introduction

The *Quadratic Assignment Problem* (QAP) in Koopmans-Beckmann form [1] is a well-studied and extremely difficult problem in combinatorial optimization. Its input consists of two $n \times n$ square matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ with real entries. (Note: all matrices in this paper have non-negative entries). The goal is to find a permutation π from the set S_n of permutations of $\{1, 2, \ldots, n\}$, that maximizes (max-QAP) or minimizes (min-QAP) the objective function

$$Z_\pi(A, B) \;:=\; \sum_{i=1}^{n}\sum_{j=1}^{n} a_{\pi(i),\pi(j)}\, b_{i,j}. \tag{1}$$

We refer the reader to the excellent book [2] by Çela for more information on the QAP and on its central role in discrete optimization.

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One branch of QAP research analyzes the algorithmic behavior of strongly-structured special cases; see for instance Çela & al [3] and Deineko & Woeginger [4] for some typical results in this direction. Burkard & al [5] state one of the most general positive results in this area which unifies a number of older results from the literature (for the exact statement, see Proposition 2 in Section 3 below). The result covers a theorem of Hardy, Littlewood & Pólya [6, 7] derived in 1926 by tedious case analysis, and it generalizes a theorem of Supnick [8] for the TSP that is derived by a lengthy sequence of meticulous exchange arguments.

Results of this paper. We present a slight generalization of the main result of Burkard & al [5]. Our proof method is essentially the same as the one in [5], and our result strongly relies and builds on the results in [5].

Our research was motivated by a recent paper of Wei & al [9] on the scheduling problem of maximizing the variance of job completion times on a single machine. The scheduling problem studied in [9] is closely related to the QAPs studied in [5], but does not quite fit into the framework. We stretch and extend the framework a little bit, so that it also covers the scheduling problem, and thereby we derive the main result of [9] as a corollary.

The paper is organized as follows. Section 2 summarizes the results of Burkard & al [5], and Section 3 states and proves our generalization. Section 4 defines the scheduling problem discussed by Wei & al [9]. Section 5 shows how this scheduling problem fits into our framework, and gives a short new proof for it.

2. An old special case of the QAP

In this section we summarize notations, definitions, and results from the QAP literature that we will apply and generalize in the main part of the paper.

An $n \times n$ matrix $A = (a_{i,j})$ is called *monotone* if $a_{i,j} \le a_{i,j+1}$ and $a_{i,j} \le a_{i+1,j}$ holds for all i, j, that is, if the entries in every row and in every column are in non-decreasing order. Matrix A is a symmetric *product matrix*, if there are non-negative real numbers $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_n$ such that

$$a_{i,j} \;=\; \alpha_i\,\alpha_j \qquad \text{for } 1 \le i, j \le n. \tag{2}$$

For two given natural numbers p, q with $p, q \le n$ the $n \times n$ *LR-block matrix* $R^{(pq)} = (r^{(pq)}_{i,j})$ has a $p \times q$ block of 1-entries in the lower right corner and 0-entries elsewhere. Formally, in such a matrix we have $r^{(pq)}_{i,j} = 1$ for all i, j with $i \ge n - p + 1$ and $j \ge n - q + 1$, and $r^{(pq)}_{i,j} = 0$ otherwise. Matrix A is an *anti-Monge matrix* (see Burkard, Klinz & Rudolf [10]), if it satisfies the so-called anti-Monge inequalities

$$a_{i,j} + a_{r,s} \;\ge\; a_{i,s} + a_{r,j}\,, \qquad \text{for all } 1 \le i < r \le n \text{ and } 1 \le j < s \le n. \tag{3}$$

It is well-known (and easy to see) that all product matrices and all LR-block matrices are special cases of monotone anti-Monge matrices. Furthermore the monotone anti-Monge matrices form a cone. The following proposition states that the LR-block matrices constitute the extremal rays of this cone.

Proposition 1. *(Rudolf & Woeginger [11]; Burkard & al [5])*
Matrix A is a monotone anti-Monge matrix, if and only if A can be written as a non-negative linear combination of LR-block matrices. □

An $n \times n$ matrix $B = (b_{i,j})$ is a *Toeplitz matrix*, if there exists a function $f\colon \{-n+1,\dots,n-1\} \to \mathbb{R}$ such that $b_{i,j} = f(i-j)$ for $1 \le i,j \le n$. The Toeplitz matrix B is said to be *generated* by function f. Note that a Toeplitz matrix is completely determined if its first row and first column are known, and that the function f essentially contains this information. Function f is *benevolent* if it satisfies the following three properties.

$$f(i) = f(-i) \qquad \text{for } 1 \le i \le n-1 \tag{4a}$$

$$f(i) \ge f(i+1) \qquad \text{for } 1 \le i \le \lfloor \tfrac{n}{2} \rfloor - 1 \tag{4b}$$

$$f(i) \ge f(n-i) \qquad \text{for } 1 \le i \le \lceil \tfrac{n}{2} \rceil - 1 \tag{4c}$$

A Toeplitz matrix is *benevolent* if it is generated by a benevolent function.

The so-called *Supnick permutation* $\pi_n^* \in S_n$ shows up in the work of Fred Supnick [8]. It is defined by $\pi_n^*(i) = 2i-1$ for $1 \le i \le \lceil \frac{n}{2} \rceil$, and $\pi_n^*(n+1-i) = 2i$ for $1 \le i \le \lfloor \frac{n}{2} \rfloor$. To keep our notation simple, we will often suppress the dependence on n and write π^* short for π_n^*. Note that π^* starts with the odd numbers in increasing order, followed by the even numbers in decreasing order. Adopting the notation $\phi = \langle \phi(1), \phi(2), \dots, \phi(n) \rangle$ for permutations ϕ, we can write $\pi^* = \langle 1, 3, 5, 7, 9, \dots, 10, 8, 6, 4, 2 \rangle$.

Proposition 2. *(Burkard & al [5])*
If A is a monotone anti-Monge matrix and B is a benevolent Toeplitz matrix, then the max-QAP *is solved to optimality by permutation π^*.* □

3. A new special case of the QAP

For an $n \times n$ matrix $B = (b_{i,j})$, we define the onion matrix $\textsc{Onion}(B) = (o_{i,j})$ as the $(n+2) \times (n+2)$ matrix that results from B by attaching a new first row, a new last row, a new first column, and a new last column, and by setting all new entries to 0. Formally,

$$o_{i,j} = \begin{cases} b_{i-1,j-1} & \text{if } 2 \le i,j \le n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, for every integer $k \ge 0$ we define the matrix k-$\textsc{Onion}(B)$ as follows. For $k = 0$, we let 0-$\textsc{Onion}(B) = B$. For $k \ge 1$, we define k-$\textsc{Onion}(B)$ as the onion matrix of $(k-1)$-$\textsc{Onion}(B)$. In other words, the $(n+2k) \times (n+2k)$ matrix k-$\textsc{Onion}(B)$ results by surrounding matrix B by k new layers (or skins) that consist entirely of zeroes. Finally, for a matrix class $\mathcal{B}$ we define $\textsc{Onion}(\mathcal{B})$ as the set of all matrices k-$\textsc{Onion}(B)$ with $B \in \mathcal{B}$ and $k \geq 0$. The onion cone for $\mathcal{B}$ is denoted $\textsc{OnionCone}(\mathcal{B})$ and contains all the non-negative linear combinations of (equi-dimensional) matrices in $\textsc{Onion}(\mathcal{B})$.

The following observation is the central tool for getting our results. (We stress once again that all matrices in this paper have non-negative entries.)

Lemma 3. *Let B be an $n \times n$ matrix. Assume that for every $n \times n$ monotone anti-Monge matrix A', the* max-QAP *for A' and B is solved to optimality by permutation π^*_n.*

Then for every $(n+2) \times (n+2)$ monotone anti-Monge matrix A, the max-QAP *for A and $\textsc{Onion}(B)$ is solved to optimality by permutation π^*_{n+2}.*

Proof. We use the cone characterization of monotone anti-Monge matrices in Proposition 1. Hence let p and q be integers with $0 \leq p, q \leq n+2$, and consider the $(n+2)\times(n+2)$ LR-block matrix $R^{(pq)}$. Let R' denote the $n \times n$ matrix that results by deleting the first two rows and columns from $R^{(pq)}$.

We first investigate the max-QAP for $R^{(pq)}$ and $\textsc{Onion}(B)$. Since $R^{(pq)}$ is a monotone matrix, its first two rows are dominated componentwise by all its other rows, and its first two columns are dominated componentwise by all its other columns. A straightforward exchange argument shows that there exists an optimal solution for the max-QAP that matches the first row and column of $R^{(pq)}$ with the first (all-zero) row and column of $\textsc{Onion}(B)$ and that matches the second row and column of $R^{(pq)}$ with the last (all-zero) row and column of $\textsc{Onion}(B)$. The contribution of these rows and columns to the objective function equals 0, and hence we may as well remove them and concentrate on the remaining n-dimensional subproblem of the max-QAP with the smaller matrices R' and B. The condition in the statement of the lemma ensures that this n-dimensional max-QAP is solved to optimality by permutation π^*_n. By reattaching the removed rows and columns, we arrive at permutation π^*_{n+2} as an optimal permutation for the max-QAP with matrices $R^{(pq)}$ and $\textsc{Onion}(B)$.

Summarizing, for all extreme rays $R^{(pq)}$ of the $(n+2)$-dimensional monotone anti-Monge matrix cone, the max-QAP for $R^{(pq)}$ and $\textsc{Onion}(B)$ is solved to optimality by permutation π^*_{n+2}. This implies an analogous statement for all the matrices in the cone, and thus proves the lemma. ☐

In our next step, we apply Lemma 3 repeatedly, and attach further skins with zeroes to matrix B. This yields under the same assumptions as in the lemma that for every $(n+2k) \times (n+2k)$ monotone anti-Monge matrix A, the max-QAP for matrix A and for k-$\textsc{Onion}(B)$ is solved to optimality by permutation π^*_{n+2k}. We summarize these observations in the following theorem.

Theorem 4. *Let $\mathcal{B}$ be a set of matrices, such that for every monotone $n \times n$ anti-Monge matrix A and for every $n \times n$ matrix $B \in \mathcal{B}$, the* max-QAP *is solved to optimality by permutation π^*. Then for every monotone anti-Monge matrix A and for every $B \in \textsc{OnionCone}(\mathcal{B})$, the* max-QAP *is also solved to optimality by permutation π^*.* ☐

With the help of Theorem 4, we can finally extend the result in Proposition 2 to the following statement.

Theorem 5. *Let A be a monotone anti-Monge matrix, and let B be a matrix in the onion cone of the set of benevolent Toeplitz matrices. Then the* max-QAP *for A and B is solved to optimality by permutation π^*.* □

4. Optimizing the variance of job completion times

Consider n jobs $J_1, \ldots, J_n$ with processing times $0 \le p_1 \le p_2 \le \cdots \le p_n$, that are to be processed without preemption and without intermediate idle time on a single machine. In a schedule σ, we denote by C_j the completion time of job J_j and we denote by $\overline{C} = \frac{1}{n}\sum_{j=1}^{n} C_j$ the average completion time of the jobs. The variance of the job completion times for schedule σ is then given as

$$\textsc{Var}(\sigma) \;=\; \sum_{j=1}^{n}(C_j - \overline{C})^2 \;= \sum_{j=1}^{n} C_j^2 - \frac{1}{n}(\sum_{j=1}^{n} C_j)^2. \qquad (5)$$

We consider (i) the problem min-CTV of minimizing this variance and (ii) the problem max-CTV of maximizing the variance (in both problems, CTV stands for completion time variance). In many aspects, problems min-CTV and max-CTV behave in a diametrical fashion. For example (Eilon & Chowdhury [12]) optimal schedules for min-CTV always are *V-shaped*: the processing times of the jobs processed before $\overline{C}$ must be in non-increasing order, and the processing times of the jobs processed after $\overline{C}$ must be in non-decreasing order. In strong contrast to this, optimal schedules for max-CTV always are *pyramidal*: the processing times of the jobs processed before $\overline{C}$ must be in non-decreasing order, and the processing times of the jobs processed after $\overline{C}$ must be in non-increasing order. (The pyramidality follows by a minor modification of the argument in [12].) As another example, any optimal schedule for min-CTV must start with the *longest* job (Schrage [13]), whereas any optimal schedule for max-CTV must start with the *shortest* job.

The scheduling literature only deals with minimizing the completion time variance. This performance measure is motivated by real-life scenarios in computer organization where it is important to provide uniform response times to the users. Problem min-CTV has been carefully investigated: it is known to be NP-complete in the ordinary sense (Kubiak [14]), it is solvable in pseudo-polynomial time (De, Ghosh & Wells [15]), and it allows a fully polynomial time approximation scheme [15]. The maximization problem max-CTV is investigated in a recent paper by Wei & al [9].

Theorem 6. *(Wei & al [9])*
Every instance of max-CTV *is solved to optimality by arranging the jobs in order of the Supnick permutation $\pi^* = \langle 1, 3, 5, 7, 9, \ldots, 10, 8, 6, 4, 2\rangle$ or in order of the permutation*

$$\tau^* \;=\; \langle 1, 2, 4, 6, 8, 10, \ldots, 9, 7, 5, 3\rangle.$$

Wei & al [9] establish Theorem 6 through a long sequence of sophisticated exchange arguments. The argument is surprisingly delicate and takes more than 12 pages. In the following section, we will provide a fairly short and simple proof for Theorem 6 that is based on the results of Section 3.

5. Job completion time variance as a QAP

In this section, we will first show that min-CTV and max-CTV are highly structured quadratic assignment problems and then we will deduce Theorem 6 from Theorem 5. We start by rewriting the objective value in (5) under the identity permutation id that runs the jobs in their natural order $J_1, \ldots, J_n$. Then $C_j = \sum_{k=1}^{j} p_k$, and we derive

$$\begin{aligned}\sum_{j=1}^{n} C_j^2 &= \sum_{j=1}^{n}(\sum_{k=1}^{j} p_k)^2 = \sum_{j=1}^{n}(\sum_{k=1}^{j} p_k^2 + 2\sum_{k=1}^{j}\sum_{i=k+1}^{j} p_k p_i) = \\ &= \sum_{j=1}^{n}(n-j+1)\, p_j^2 \;+\; 2\sum_{1\le i<j\le n}(n-j+1)\, p_i p_j. \end{aligned} \tag{6}$$

Furthermore we determine

$$\begin{aligned}(\sum_{j=1}^{n} C_j)^2 &= (\sum_{j=1}^{n}(n-j+1)\, p_j)^2 = \\ &= \sum_{j=1}^{n}(n-j+1)^2\, p_j^2 \;+\; 2\sum_{1\le i<j\le n}(n-i+1)(n-j+1)\, p_i p_j. \end{aligned} \tag{7}$$

Next, we multiply the objective value in (5) by n and then rewrite it with the help of (6) and (7) in the form

$$\begin{aligned} n\cdot \mathrm{VAR}(id) &= n\sum_{j=1}^{n} C_j^2 - (\sum_{j=1}^{n} C_j)^2 = \\ &= \sum_{j=1}^{n}(j-1)(n-j+1)\, p_j^2 \;+\; 2\sum_{1\le i<j\le n}(i-1)(n-j+1)\, p_i p_j. \end{aligned} \tag{8}$$

For an arbitrary schedule σ, we simply replace every occurrence of p_j in (8) by $p_{\sigma(i)}$. Straightforward algebraic manipulations then lead to

$$n\cdot \mathrm{VAR}(\sigma) = \sum_{i=1}^{n}\sum_{j=1}^{n} p_{\sigma(i)} p_{\sigma(j)}\, (\min\{i,j\}-1)\, (n+1-\max\{i,j\}). \tag{9}$$

Note that the right hand side in (9) exactly is a QAP in Koopmans-Beckmann form as formulated in (1).

$$B_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 4 & 3 & 2 & 1 \\ 0 & 4 & 8 & 6 & 4 & 2 \\ 0 & 3 & 6 & 9 & 6 & 3 \\ 0 & 2 & 4 & 6 & 8 & 4 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix} \qquad B_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 5 & 4 & 3 & 2 & 1 \\ 0 & 5 & 10 & 8 & 6 & 4 & 2 \\ 0 & 4 & 8 & 12 & 9 & 6 & 3 \\ 0 & 3 & 6 & 9 & 12 & 8 & 4 \\ 0 & 2 & 4 & 6 & 8 & 10 & 5 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

Figure 1: The matrices B_6 and B_7.

Lemma 7. *The scheduling problems* min-CTV *and* max-CTV *can be formulated as* min-QAP *and* max-QAP*, respectively. The underlying $n \times n$ matrix A is a symmetric product matrix as defined in (2), with non-negative real numbers $p_1 \le p_2 \le \cdots \le p_n$ and*

$$a_{i,j} \;=\; p_i\, p_j \qquad \text{for } 1 \le i, j \le n.$$

The underlying $n \times n$ matrix $B_n = (b_{i,j})$ is given by

$$b_{i,j} \;=\; (\min\{i,j\} - 1)\,(n + 1 - \max\{i,j\}) \qquad \text{for } 1 \le i, j \le n.$$

Figure 1 shows matrices B_6 and B_7 as an illustration. The first row and column of such an $(n+1)\times(n+1)$ matrix B_{n+1} entirely consist of zeroes, whereas all its remaining entries are strictly positive and form the $n \times n$ matrix $B_n^* = (b_{i,j}^*)$ with

$$b_{i,j}^* \;=\; \min\{i,j\}\,(n + 1 - \max\{i,j\}) \qquad \text{for } 1 \le i, j \le n. \tag{10}$$

Matrix B_n^* is not only symmetric with respect to its main diagonal, but it is also symmetric with respect to its counter-diagonal:

$$b_{i,j}^* \;=\; b_{n-j+1,n-i+1}^* \qquad \text{for } 1 \le i, j \le n. \tag{11}$$

Furthermore, matrix B_n^* fits into our framework developed in Section 3:

Lemma 8. *The $n \times n$ matrix B_n^* as defined in (10) belongs to the onion cone of the set of benevolent Toeplitz matrices.*

Proof. We introduce the $n \times n$ Toeplitz matrix $T_n = (t_{i,j})$ with $t_{i,j} = n - |i - j|$. Since the generating function $f(x) = n - |x|$ of T_n satisfies conditions (4a)–(4c), matrix T_n is a benevolent Toeplitz matrix. We claim that for $n \ge 2$

$$B_n^* \;=\; T_n + \textsc{Onion}(B_{n-2}^*). \tag{12}$$

Since all involved matrices are symmetric, it is sufficient to consider the entry in some fixed row i and some fixed column j with $i \le j$. Then the corresponding entry in

Onion(B^*_{n-2}) equals $(i-1)(n-j)$, and the corresponding entry in matrix T_n is $t_{i,j} = n+i-j$. Since the sum of these two entries is $i\,(n+1-j)$ and thus coincides with entry $b^*_{i,j}$ in B^*_n, the claim is proved.

Now a simple induction completes the proof of the lemma. The cases $n = 1$ and $n = 2$ are straightforward, and (12) yields the inductive step. □

Now recall from Section 4 that any optimal schedule for min-CTV/max-CTV must start with the longest/shortest job in the instance. This is also easily observed from the structure of matrix B_n: the QAP must match the first (all-zero) row and column of matrix B_n with the heaviest/lightest row and column of the product matrix A. Hence let us remove the first row and column from matrix B_n, and let us remove the row and column for the longest/shortest job from matrix A. For min-CTV, this is already the end of the story line.

For max-CTV, we are left with the $(n-1)\times(n-1)$ product matrix for $p_2 \le p_3 \le \cdots \le p_n$ and with the $(n-1)\times(n-1)$ matrix B^*_{n-1}. Since the product matrix is a monotone anti-Monge matrix and since B^*_{n-1} is in the onion cone of the set of benevolent Toeplitz matrices (Lemma 8), Theorem 5 applies and yields the Supnick permutation as optimal solution for the corresponding max-QAP. The corresponding schedule for the $n-1$ jobs $J_2, J_3, \ldots, J_n$ is

$$\langle 2, 4, 6, 8, 10, \ldots, 9, 7, 5, 3\rangle.$$

Since matrix B^*_{n-1} is symmetric with respect to its counter-diagonal as shown in (11), also the left-right reflected mirror image of the Supnick permutation is an optimal solution for the max-QAP. The corresponding schedule for the $n-1$ jobs $J_2, J_3, \ldots, J_n$ is

$$\langle 3, 5, 7, 9, \ldots, 10, 8, 6, 4, 2\rangle.$$

By putting the removed job J_1 at the beginning of these two schedules, we arrive at the two permutations π^* and τ^* from Theorem 6 as optimal permutations for max-CTV.

And this completes our new proof of Theorem 6.

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