Quadratic and Three Dimensional Assignments: An Annotated Bibliography *

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Abstract

The amount of literature on quadratic assignment and related problems has already grown so much that overviewing it to determine the most relevant developments and the most recent trends becomes more and more difficult. This paper provides a collection of references on quadratic and three dimensional assignment problems together with brief annotations. We consider all aspects of the quadratic assignment problem (QAP) ranging form linearizations and equivalents formulations to polynomially solvable special cases and asymptotic behavior. Similarly, the most important research directions on three dimensional assignment problems (3DAP) are covered. Hereby this paper lies somewhere in the middle ground between a pure bibliography and a survey article. It will soon appear as a separate chapter in "Annotated Bibliographies in Combinatorial Optimization", edited by M. Dell'Amico, F. Maffioli and S. Martello.

Concentrating on contributions which appeared in 1985 or later and focusing on the most recent results, we consider, however, also seminal work on the roots of the problems at hand and survey papers which can serve as precious sources of related literature.

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1 Introduction

Given two $n \times n$ matrices A and B the quadratic assignment problem (QAP) of size n can be stated as follows

$$\min_{\phi \in \mathcal{S}_n} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{\phi(i)\phi(j)} b_{ij} \,,$$

where $\mathcal{S}_n$ is the set of permutations ϕ of $\{1, 2, \ldots, n\}$. Initially the QAP arose as a mathematical model of a location problem concerning economic activities. In the context of location problems which still remain a major application of the QAP, n facilities and n locations are given. Matrix A is the flow matrix, i.e. a_{ij} is the flow of materials moving from facility i to facility j, and matrix B is the distance matrix, i.e. b_{kl} is the distance between facilities k and l. The cost of simultaneously assigning facility i to location k and facility j to location l is $a_{ij} b_{kl}$. The objective of the QAP consists of finding an assignment of the facilities to the locations with the minimum overall cost. Nowadays, a large variety of other *practical applications* of the QAP are known, including such areas as scheduling, manufacturing, parallel and distributed computing, statistical data analysis and chemistry. From the *theoretical point of view*, other combinatorial optimization and graph theoretical problems can be formulated as QAPs. Just to mention some well known examples consider the traveling salesman problem, the turbine problem, the linear ordering problem, graph partitioning problems, subgraph isomorphism and maximum clique problems. Due to its theoretical and practical relevance, but also due to its *complexity*, the QAP has been subject of extensive research since its first occurrence in 1957. In the last decade we have seen a dramatic increase of the size of NP-hard combinatorial optimization problems which can be efficiently solved in practice. Unfortunately, the QAP *is not* one of them; QAP instances of size larger than 20 are still considered intractable. Thus, the QAP still remains a challenging problem from both theoretical and practical point of view.

The research done on the QAP covers more or less all of its aspects. With the intention to identify new structural combinatorial properties a number of alternative formulations for the QAP have been given. Ranging from equivalent Boolean linear and mixed integer linear programs to the trace formulation, they have led to diverse lower bounding procedures and exact solution methods for this problem. It is probably remarkable that quite different approaches have been applied to this end: combinatorial methods, eigenvalue computation and subgradient and nonsmooth optimization techniques. The resulting lower bounds have been incorporated in cutting planes and branch and bound algorithms for the QAP, the latter being considered the most efficient. Recently, parallel implementation of branch and bound methods have enabled the solution of test instances of size 20. However, even the most sophisticated implementations of exact algorithms fail in solving real size QAPs and heuristics still remain the unique mean to solve medium to large size instances of the problem. Among the large variety of heuristics proposed for the QAP the so called metaheuristics, tabu search, simulated annealing and genetic algorithms, seem to be the most efficient. As these methods are based on neighborhood search, they are also appropriate for parallel implementations. This enables in turn the heuristic solution of real life problems. Unfortunately, there is no guarantee on the quality of the solutions produced by these methods and no tight bounds for large sized QAPs are known. This is not surprising when considering that even the approximation problem for QAPs is in general NP-hard. On the other side, under certain probabilistic conditions the random QAP becomes in some sense trivial as the size of the problem increases.

Another research direction on QAPs concerns restricted versions of the problem. Clearly, most of the efforts focus on identifying polynomially solvable cases of the QAP. However, the identification of provably NP-hard cases helps on understanding structural properties of the problem. Recently, QAPs whose coefficient matrices have a special combinatorial structure have been investigated leading to some new polynomially solvable cases. However, only a few results of this type are know and a lot remains to do in this direction.

Another object of research work related to QAPs concerns its generalizations. Given the large area of applications of this problem, its numerous generalizations and related problems should not be surprising. The generalizations may be related to the structure of the problem coefficients or to the set of feasible solutions. Two well known examples are probably the *biquadratic assignment problem* (BiQAP) and the *semi-quadratic assignment problem* (SQAP).

Another widely known and well studied assignment problem is the *multidimensional assignment problem* (MAP), in particular the *three dimensional assignment problem* (3-DAP). There are two well distinguished versions of the 3-DAP: the *axial* 3-DAP and the *planar* 3-DAP. In the 3-DAP of size n we are given three disjoint sets I, J, K of cardinality n each and a weight c_{ijk} associated with each ordered triplet $(i,j,k) \in I \times J \times K$. In the axial 3-DAP we want to find a minimum (maximum) weight collection of n pairwise disjoint triplets as above, whereas in the planar 3-DAP the goal is to find n^2 triplets forming n disjoint sets of n disjoint triplets each. The multidimensional assignment problem arises as a generalization of the axial 3-DAP when n-tuples are considered instead of triplets. Both the axial and the planar 3-DAP are known to be NP-hard and have several applications with respect to scheduling and time-tabling problems. A recent application of the MAP concerns data association problems in multitarget tracking and multisensor data fusion. The axial (planar) 3-DAP is a close relative to the (solid) transportation problem. This relationship has been probably helpful on studying the facial structure of these problems. Some classes of facet defining inequalities and corresponding separation algorithms have been derived.

Among algorithms known for 3-DAPs some branch and bound methods involving Lagrangean relaxation and subgradient optimization can be mentioned, the axial problem being the mostly studied. Recently, a tabu search algorithm for the planar 3-DAP has been proposed.

Finally, some investigations have been done on special cases of the axial 3-DAP. These investigations concern problems whose coefficients have a special structure. e.g. are decomposable or fulfill the triangle inequality, or possess Monge-like properties. It turns out that in most of the cases the problems remain NP-hard, unless their coefficients fulfill additional, more restrictive conditions. The latter lead then to polynomially solvable and polynomially approximable cases, respectively.

There exists an abundant literature on the QAP and its generalizations. In drawing up this bibliography, we have concentrated on publications that appeared in 1985 or later, focusing on the most recent contributions. However, seminal work related to the roots of the QAP or review articles which contain a large number of pointers to relevant previous work have also been mentioned. When reviewing papers related to algorithmic aspects of the problem, we have only reported on those which present the best computational results, unless relevant theoretical contribution is provided. We hope to have not overlooked any important contribution on the considered problems. However, we would be pleased to hear about additional relevant work in this area and we would highly appreciate any related pointers.

2 Books and Surveys

The following survey papers can serve as a general introduction to quadratic assignment problems. Covering all aspects of research on QAPs, these papers provide also a large number of pointers to the roots of the QAP and to earlier surveys which are not listed in this section.

G. Finke, R.E. Burkard, F. Rendl (1987). Quadratic assignment problems. S. Martello, G. Laporte, M. Minoux, C. Ribeiro (eds.). *Surveys in Combinatorial Optimization*, Ann. Discr. Math. 31, North-Holland, Amsterdam, 61–82.

This survey focuses on the "trace formulation" of the QAP. The eigenvalue approach for the lower bound computation in the case of symmetric QAPs is introduced together with a reduction scheme for improving the resulting bounds.

S.W. Hadley, F. Rendl, H. Wolkowicz (1990). Bounds for the quadratic assignment problem using continuous optimization techniques. *Integer Programming and Combinatorial Optimization*, University of Waterloo Press, 237–248.

This article reviews lower bounding procedures for the QAP based on continuous optimization techniques. Eigenvalue techniques, reduced gradient methods, trust region methods, sequential quadratic programming and subdifferential calculus are applied to approximations and relaxations of the QAP.

R.E. Burkard (1991). Locations with Spatial Interactions: The Quadratic Assignment Problem. P.B. Mirchandani, R.L. Francis (eds.). *Discrete Location Theory*, John Wiley & Sons, 387–437.

This survey resumes known results related to (mixed) integer programming formulations, bounding procedures, exact algorithms and heuristics for QAPs. Some typical applications of the QAP are described and a number of papers describing less typical applications are referenced. Moreover, the asymptotic behavior of the QAP is described.

P.M. Pardalos, F. Rendl, H. Wolkowicz (1994). The quadratic assignment problem: A survey and recent developments. *DIMACS Series Discr. Math. Theor. Comp. Sci.* 16, 1–42.

This is the most recent survey on the QAP. It appeared as an introductory article in the proceedings book of the DIMACS workshop on quadratic assignment and related problems. It focuses on recent results concerning computation of lower bounds, computational complexity, heuristic approaches for the QAP and its generalizations.

Quadratic Assignment and Related Problems, Proc. DIMACS Workshop on Quadratic Assignment Problems, P.M. Pardalos, H. Wolkowicz (eds.). DIMACS Series Discr. Math. Theor. Comp. Sci. 16.

This book offers a collection of up-to-date contributions on computational approaches to the QAP and its applications.

3 Roots of the Quadratic Assignment Problem (QAP): Basic Facts and Complexity

T.C. Koopmans, M.J. Beckmann (1957). Assignment problems and the location of economic activities. *Econometrica* 25, 53–76.

This is the first occurrence of the standard QAP formulation: $\min_\phi \sum_{i,j=1}^{n} a_{\phi(i)\phi(j)} b_{ij}$, where ϕ ranges over all permutations of $\{1, 2, \ldots, n\}$. The QAP is derived as a mathematical formulation of a problem arising along with the location of economic activities.

P.C. Gilmore (1962). Optimal and suboptimal algorithms for the quadratic assignment problem. *SIAM J. Appl. Math.* 10, 305–313.

This paper introduces the so called *Gilmore-Lawler bounds* which still remain one of the most important and frequently used bounds for the QAP. Based on these bounds, two heuristic approaches are proposed.

E.L. Lawler (1963). The quadratic assignment problem. *Management Sci.* 9, 586–599.

A more general QAP is introduced, where the objective is the minimization of a double sum of the form $\sum_{i,j=1}^{n} d_{\pi(i)\pi(j)ij}$ over all permutations ϕ of $\{1, 2, \ldots, n\}$. The problem coefficients d_{ijkl} form an array with n^4 elements. Moreover, the author derives an equivalent integer programming formulation for this problem and describes the computation of lower bounds.

C.E. Nugent, T.E. Vollmann, J. Ruml (1968). An experimental comparison of techniques for the assignment of facilities to locations, *Oper. Res.* 16, 150–173.

An improvement method combined with random elements is proposed. This method is compared with deterministic improvement algorithms on a set of QAP test instances. Nowadays these instances are known as *Nugent's problems* and are frequently used for experimental purposes.

G.W. Graves, A.B. Whinston (1970). An algorithm for the quadratic assignment problem. *Management Sci.* 17, 453–471.

The authors derive formulas for the mean and the variance of the objective function value of the QAP. Moreover, enumerative algorithms are proposed which exploit this statistical information.

R.E. Burkard (1974). Quadratische Bottleneckprobleme. *Oper. Res. Verfahren* 18, 26–41.

The QAP with bottleneck objective function is introduced. The goal is to minimize $\max_{1\le i,j\le n} a_{\phi(i)\phi(j)} b_{ij}$ over all permutations ϕ of $\{1, 2, \ldots, n\}$. Moreover, the author proposes lower bounds for the *bottleneck QAP* to be incorporated in branch and bound algorithms.

S. Sahni, T. Gonzalez (1976). P-complete approximation problems. *J. ACM* 23, 555–565.

The computational complexity of the QAP is investigated showing that this problem is strongly NP-hard. Moreover, it is shown that the existence of a polynomial ϵ-approximate algorithm for QAPs implies $\mathcal{P} = \mathcal{NP}$.

M. Queyranne (1986). Performance ratio of polynomial heuristics for triangle inequality quadratic assignment problems. *Oper. Res. Lett.* 4, 231–234.

The author considers QAPs with coefficient matrices fulfilling the triangle inequality. It is shown that for such QAPs no polynomial heuristic algorithm with bounded asymptotic performance ratio exists unless $\mathcal{P} = \mathcal{NP}$.

K.A. Murthy, P. Pardalos and Y. Li (1992). A local search algorithm for the quadratic assignment problem. *Informatica* 3, 594-538.

A new neighborhood for QAPs is proposed which is similar to the Kernighan-Lin neighborhood for the graph partitioning problem. It is shown that the corresponding local search problem is PLS-complete.

4 Linearizations of the QAP

The QAP can be equivalently formulated as a Boolean, an integer or a mixed integer linear program. There exists a large number of such equivalent formulations for the QAP. This approach is particularly fruitful concerning the computation of lower bounds.

L. Kaufman, F. Broeckx (1978). An algorithm for the quadratic assignment problem using Benders' decomposition. *European J. Oper. Res.* 2, 204–211.

The authors propose an equivalent formulation for the QAP as a mixed integer linear program with n^2 real variables, n^2 integer variables and $O(n^2)$ constraints. This is one of the "smallest" linearizations of the QAP with respect to the number of variables and constraints.

M.S. Bazaraa, H.D. Sherali (1980). Benders' partitioning scheme applied to a new formulation of the quadratic assignment problem. *Naval Res. Log. Quart.* 27, 29–41.

An equivalent formulation of the QAP as a mixed integer linear program with a highly specialized structure is proposed. This formulation which involves n^2 Boolean variables, $n^2(n-1)/2$ real variables and $2n^2$ linear constraints, permits the effective use of the partitioning scheme of Benders.

W.P. Adams, H.D.Sherali (1986). A tight linearization and an algorithm for zero-one quadratic programming problems. *Management Sci.* 32, 1274–1290.

A linearization for a class of linearly constrained $0-1$ quadratic programming problems containing the QAP is proposed. It is shown that this linearization is tighter than other ones existing in the literature. Moreover, an implicit enumeration algorithm which makes use of the strength of this linearization is derived.

W.P. Adams, T.A. Johnson (1994). Improved linear programming-based lower bounds for the quadratic assignment problem. *DIMACS Series Discr. Math. Theor. Comp. Sci.* 16, 43–76.

A new mixed 0-1 linear formulation for the QAP is proposed. By appropriately surrogating selected constraints and combining variables, most of the known linear formulations for the QAP can be obtained. Moreover, most of the resulting bounding techniques can be described in terms of the Lagrangean dual of this new formulation of the QAP. A dual-ascent procedure is proposed for suboptimally solving a relaxation of the dual of the new QAP formulation deriving also new lower bounds.

5 Lower Bounds for the QAP

N. Christofides, M. Gerrard (1981). A graph theoretic analysis of bounds for the quadratic assignment problem. P.Hansen (ed.). *Studies on Graphs and Discrete Programming*, North-Holland, 61–68.

The authors consider a special version of the QAP, where the feasible solutions correspond to isomorphisms of graphs. This version of the QAP is polynomially solvable in the case that the coefficient matrices are weighted adjacency matrices of isomorphic trees or other simple graphs, eg. wheels or cycles. The latter solvable cases can be used to generate lower bounds for the general QAP.

The following three papers deal with Lagrangean techniques for computing lower bounds.

A.M. Frieze, J. Yadegar (1983). On the quadratic assignment problem. *Discr. Appl. Math.* 5, 89–98.

The relationship between the Gilmore-Lawler bounds for the QAP on reduced matrices and a Lagrangean relaxation of a particular mixed $0-1$ linear formulation for the QAP is investigated. The Gilmore-Lawler bounds obtained by involving an "optimal" reduction are dominated by the continuous relaxation of the proposed linear formulation for the QAP.

A.A. Assad, W. Xu (1985). On lower bounds for a class of quadratic $0,1$ programs. *Oper. Res. Lett.* 4, 175–180.

P. Carraresi, F. Malucelli (1992). A new lower bound for the quadratic assignment problem. *Oper. Res.* 40, Suppl. No. **1**, S22–S27.

In these two papers iterative methods are used for generating non-decreasing sequences of lower bounds for the QAP. In each iteration the problem is reformulated and a lower bound for the new formulation is computed. The reformulation is based on a Lagrangean dual-ascent procedure and on the information given by the dual variables arising along with the lower bound computation, respectively.

The following four papers use the trace formulation of the QAP for generating lower bounds based on eigenvalue computations.

F. Rendl (1985). Ranking scalar products to improve bounds for the quadratic assignment problem. *European J. Oper. Res.* 20, 363–372.

The author reconsiders the eigenvalue bounds for QAPs proposed by Finke, Burkard and Rendl (1987) as described in Section 2. In the case when the linear term resulting after the reduction mostly influences the objective function, the eigenvalue bound can be improved by ranking the k-best solutions of the linear term.

S.W. Hadley, F. Rendl, H. Wolkowicz (1992). Symmetrization of nonsymmetric quadratic assignment problems and the Hoffman-Wielandt inequality, *Linear Alg. Appl.* 167, 53–64.

A technique is proposed to transform a nonsymmetric QAP into an equivalent QAP on Hermitian matrices. The eigenvalue bound for symmetric QAPs is extended to the general problem and Hoffman-Wielandt-type eigenvalue inequalities for general matrices are derived.

F. Rendl, H. Wolkowicz (1992). Applications of parametric programming and eigenvalue maximization to the quadratic assignment problem. *Math. Program.* 53, 63-78.

The classical eigenvalue bounds for QAPs on symmetric matrices can be improved by applying special reduction schemes. The authors derive an "optimal" reduction taking simultaneously into account the quadratic term and the linear term of the objective function. This involves a steepest-ascent algorithm based on subdifferential calculus.

S.W. Hadley, F. Rendl, H. Wolkowicz (1993). A new lower bound via projection for the quadratic assignment problem. *Math. Oper. Res.* 17, 727–739.

The standard eigenvalue bounds for the QAP are improved. The new bounds make use of a tighter relaxation on orthogonal matrices with constant row and column sums. The additional constraints of the new relaxation are projected into the space of orthogonal matrices of size $n-1$, where n is the size of the given QAP. For bounding the quadratic part of the projected program standard eigenvalue approaches are used.

S.E. Karisch, F. Rendl (1995). Lower bounds for the quadratic assignment problem via triangle decompositions. *Math. Program.* 71, 137-151.

QAPs where one of the coefficient matrices is the distance matrix of a grid graph are considered. The problem is decomposed into a trivially solvable QAP and a so called "residual QAP". A lower bound for the residual problem is computed via projection and nonsmooth optimization techniques are used to derive an appropriate decomposition.

6 Exact Algorithms for the QAP

A large variety of exact algorithms has been proposed for the QAP among which the branch and bound methods generally yield the better results. Many of these algorithms are mentioned and/or described in the surveys cited in Section 2. A newer development is derived by Edwards in the following paper.

C.S. Edwards (1980). A branch and bound algorithm for the Koopmans-Beckmann quadratic assignment problem. *Math. Program. Study* 13, 35–52.

The branch and bound approach is based on the trace formulation of the QAP which allows to effectively use a binary branching rule.

The performance of branch and bound algorithms depends significantly on the efficience and on the quality of the involved lower bounds. The performance of such algorithms can be also improved by a smart use of the available hardware in parallel implementations. The following three papers describe some parallel branch and bound algorithms for the QAP.

C. Roucairol (1987). A parallel branch and bound algorithm for the quadratic assignment problem. *Discr. Appl. Math.* 18, 221–225.
P. Pardalos, J. Crouse (1989). A parallel algorithm for the quadratic assignment problem. *Proc. Supercomputing Conf.*, 351–360.
J. Clausen, M. Perregård, Solving large quadratic assignment problems in parallel. *Computational Opt. Appl.* (to appear)

T. Mautor, C. Roucairol (1994). A new exact algorithm for the solution of quadratic assignment problems, *Discr. Appl. Math.* 55, 281–293.

It is shown how to exploit the symmetries on the cost matrix in order to reduce the branch and bound tree. The proposed branch and bound algorithm uses polytomic branching rule and outperforms most of the other branch and bound schemes existing at that time.

N. Christofides, E. Benavent (1989). An exact algorithm for the quadratic assignment problem on a tree. *Oper. Res.* 37, 760–768.

A special case of the QAP is considered, where the flow matrix is the weighted adjacency matrix of a tree. A branch and bound method for this NP-hard special case is derived. An integer programming formulation for this problem is given and its Lagrangean relaxation is solved by using a dynamic programming scheme. This approach produces tight lower bounds.

M.E. Dyer, A.M. Frieze, C.J.H. McDiarmid (1986). On linear programs with random costs. *Math. Program.* 35, 3–16.

The authors derive an interesting result on the size of branch and bound trees for random QAPs. The considered lower bounds arise as solutions of an LP relaxation of the Boolean linear programming formulation of the QAP given by Frieze and Yadegar (1983) and described in Section 5). It is shown that in case of binary branching the number of the explored branching nodes grows super-exponentially with probability tending to 1 as the site of the problem approaches infinity.

7 Heuristics for the QAP

There is a large variety of heuristics for the QAP ranging from construction and deterministic improvement methods to tabu search and simulation based algorithms. In the following we will only mention some of the most recent approaches.

Y. Li, P.M. Pardalos, M. Resende (1994). A greedy randomized adaptive search procedure for the quadratic assignment problem. *DIMACS Series Discr. Math. Theor. Comp. Sci.* 16, 237–261.

The authors propose an improvement method which combines greedy elements with probabilistic aspects. The so called GRASP shows a good computational behavior in many QAP instances from QAPLIB (see Section 10 below).

7.1 Simulated annealing approaches

R.E. Burkard, F. Rendl (1983). A thermodynamically motivated simulation procedure for combinatorial optimization problems. *European J. Oper. Res.* 17, 169–174.

This is one of the first applications of *simulated annealing* (SA) to the QAP. It is shown that SA outperforms most of the existing heuristics for the QAP at that time.

M.R. Wilhelm, T.L. Ward (1987). Solving quadratic assignment problems by "simulated annealing". *IIE Trans.* 19, 107-119.

SA is improved by introducing "equilibria" components which comply with the statistic mechanics background of the underlying Metropolis' algorithm.

D.T. Connolly (1990). An improved annealing scheme for the QAP. *European J. Oper. Res.* 46, 93–100.

A new element of the annealing scheme, the so called *optimal temperature*, is introduced. The corresponding algorithm yields a promising improvement of the trade-off between computation time and solution quality.

7.2 Tabu search approaches

One of the first applications of tabu search to the QAP and a parallel implementation of tabu search for QAPs can be found in the following two papers, respectively.

J. Skorin-Kapov (1990). Tabu search applied to the quadratic assignment problem. *ORSA J. Comput.* 2, 33–45.
J. Chakrapani, J. Skorin-Kapov (1993). Massively parallel tabu search for the quadratic assignment problem. *Ann. Oper. Res.* 41, 327–342.

The performance of tabu search algorithms depends very much on the size of the tabu list and on the way this list is handled. Two of the most effective strategies leading to a good trade-off between the diversification and the intensification of the search are presented in the following two papers.

E. Taillard (1990). Robust taboo search for the quadratic assignment problem. *Parallel Comput.* 17, 443–455.
R. Battiti, G. Tecchiolli (1994). The reactive tabu search. *ORSA J. Comput.* 6, 126–140.

7.3 Genetic algorithms

We believe that among genetic approaches for QAPs, the following is a remarkable contribution.

R.K. Ahuja, J.B. Orlin, A. Tivari (1995). A greedy genetic algorithm for the quadratic assignment problem. *Working paper*, Sloan School of Management, MIT.

This genetic algorithm attempts to strike a balance between diversity and a bias towards fitter individuals. Appropriate greedy elements are combined to this end with genetic ingredients like new crossover schemes, tournamenting, periodic local optimization and an immigration rule that promotes diversity.

8 Asymptotic Behavior of the QAP

Under natural probabilistic constraints on the input data, the QAP show an interesting asymptotic behavior. Namely, the ratio between the "best" and the "worst" value of the objective function approaches 1 with probability tending to 1 as the size of the problem approaches infinity. This behavior was first shown by Burkard and Fincke for sum and bottleneck objectives:

R.E. Burkard, U. Fincke (1983). The asymptotic probabilistic behavior of quadratic sum assignment problems. *Z. Oper. Res.* 27, 73–81.
R.E. Burkard, U. Fincke (1982). On random quadratic bottleneck assignment problems. *Math. Program.* 23, 227–232.

J.B.G. Frenk, M. van Houweninge, A.H.G. Rinnooy Kan (1985). Asymptotic properties of the quadratic assignment problem. *Math. Oper. Res.* 10, 100-116.

The range of the convergence in the above mentioned behavior is improved from "with probability" to "almost sure".

W.T. Rhee (1988). A note on asymptotic properties of the quadratic assignment problem. *Oper. Res. Lett.* 7, 197–200.

The results of Frenk, Houweninge and Rinnooy Kan (1988) are improved by providing a simpler proof and sharper estimations for the almost sure convergence in the asymptotic behavior of the QAP.

W.T. Rhee (1988). Stochastic analysis of the quadratic assignment problem. *Math. Oper. Res.* 16, 223–239.

The maximization version of the QAP is considered. A simple greedy approach is described which produces a good approximation of the optimal solution with overhelming probability. This complies with the previous results on the asymptotic behavior of the QAP.

E. Bonomi, J.-L. Lutton (1986). The asymptotic behavior of quadratic sum assignment problems: A statistical mechanics approach. *European J. Oper. Res.* 26, 295–300.

A statistical mechanics approach, based on the Boltzmann distribution and the Metropolis algorithm, is applied to study the asymptotic behavior of the QAP. The authors derive in this way the same result as Burkard and Fincke (1983) and perform numerical experiments which confirm this theoretical result.

In the following two papers a combinatorial condition which guarantees an analogous asymptotic behavior for general combinatorial optimization problems is singled out. The first paper shows convergence with probability whereas in the second one an almost sure convergence is proven.

R.E. Burkard, U. Fincke (1985). Probabilistic asymptotic properties of some combinatorial optimization problems. *Discr. Appl. Math.* 12, 21–29.
W. Szpankowski (1995). Combinatorial optimization problems for which almost every algorithm is asymptotically optimal!. *Optimization* 33, 359–367.

9 Polynomially Solvable Cases of the QAP

The first polynomially solvable cases of the QAP were identified by Christofides and Gerrard (1976) as mentioned in Section 5. These results are extended and generalized to minimal vertex series-parallel (MVSP) digraphs as described in the following paper.

F. Rendl (1986). Quadratic assignment problems on series parallel digraphs. *Z. Oper. Res.* 30, 161–173.

It is shown that the general version of the QAP on isomorphic MVSPs is NP-hard. However, MVSP digraphs which do not contain the bipartite digraph $K_{2,2}$ as vertex induced subgraph lead to polynomially solvable cases. A polynomial time algorithm is proposed for solving the latter.

Recent investigation on polynomially solvable cases of the QAP rely on special combinatorial properties of the involved coefficient matrices, as shown in the following two papers.

R.E. Burkard, E. Çela, G. Rote, G.J. Woeginger (1996). The quadratic assignment problem with an anti-Monge and a Toeplitz matrix: Easy and hard cases. *Math. Program.* (to appear)
R.E. Burkard, E. Çela, V.M. Demidenko N.N. Metelski, G.J. Woeginger (1996). Easy and hard cases of the quadratic assignment problem: A survey. *Working paper*, Graz University of Technology.

10 Codes and Data for the QAP

R.E. Burkard, U. Derigs (1980). Assignment and Matching Problems: Solution Methods with FORTRAN- Programs. *Lecture Notes Econ. Math. Sys.* 184, Springer-Verlag, Berlin.

This book contains FORTRAN codes for exact and heuristic algorithms for the QAP. A pointer to the source files can be found at
`http://fmatbhp1.tu-graz.ac.at/~karisch/qaplib/`

R.E. Burkard, S.E. Karisch, F. Rendl (1991). QAPLIB - a quadratic assignment problem library, *European J. Oper. Res.* 55, 115–119. Updated version March 1996.

This paper describes a library of test instances for the QAP. For each instance the best known solutions and the corresponding objective function values are given. The updated version provides also the best known lower bounds for each test instance. This library can be found at `http://fmatbhp1.tu-graz.ac.at/~karisch/qaplib/` and is also available per anonymous ftp at `ftp.tu-graz.ac.at/pub/papers/qaplib`.

The following two papers propose algorithms for generating QAP instances with known optimal solution.

G.S. Palubeckis (1988). Generation of quadratic assignment test problems with known optimal solutions. (in Russian). *U.S.S.R. Comput. Maths. Math. Phys.* 28, 97–98.
Y. Li, P.M. Pardalos (1992). Generating quadratic assignment test problems with known optimal permutations. *Computational Opt. Appl.* 1, 163–184.

The authors show that the test instances generated by the Palubeckis algorithm are "easy" in the sense that the corresponding optimal value of the objective function can be computed in polynomial time. The proof relies on the fact that the involved coefficient matrices are Euclidean. The Palubeckis' idea is then generalized to generate test instance with known optimal solution whose coefficient matrices are not Euclidean.

11 Generalizations

A natural generalization of the QAP, the so called *biquadratic assignment problem* (BiQAP), arises in the VLSI design. The BiQAP coefficients are organized in four dimensional arrays and an instance of size n looks as follows:

$$\min_{\pi \in \mathcal{S}_n} \sum_{i,j,k,l=1}^{n} a_{\pi(i)\pi(j)\pi(k)\pi(l)} b_{ijkl}$$

where $\mathcal{S}_n$ is the set of permutations of $\{1, 2, \ldots, n\}$. The following two papers generalize previous work on QAPs to derive linearizations, lower bounds and heuristic approaches for the BiQAP. Moreover, the asymptotic behavior of the BiQAP is investigated and it is shown that it is analogous to that of the QAP.

R.E. Burkard, E. Çela, B. Klinz (1994). On the biquadratic assignment problem. *DIMACS Series Discr. Math. Theor. Comp. Sci.* 16, 117–144.
R.E. Burkard, E. Çela (1995). Heuristics for biquadratic assignment problems and their computational comparison. *European J. Oper. Res.* 83, 283–300.

A number of applications for the so called *semi-quadratic assignment problem* (SQAP) are described in the following three papers. The SQAP has the same objective function as the QAP, whereas the feasible solutions do not need to be permutations but simply injective functions mapping $\{1, 2, \ldots, n\}$ into itself. The following references provide also pointers to bounding procedures, heuristics and polynomially special cases of the SQAP.

R.J. Freeman, D.C. Gogerty, G.W. Graves, R.B.S. Brooks (1966). A mathematical model of supply support for space operations. *Oper. Res.* 14, 1–15.
V.F. Magirou, J.Z. Milis (1989). An algorithm for the multiprocessor assignment problem. *Oper. Res. Lett.* 8, 351–356.
F. Malucelli, D. Pretolani (1993). *Lower bounds for the quadratic semi-assignment problem.* Technical Report 955, Centre des Recherches sur les transports, Université de Montréal.

12 The 3-Dimensional Assignment Problem (3-DAP): Statement and Complexity

The multidimensional assignment problem (MAP) and some of its applications were introduced in:

W.P. Pierskalla (1968). The multidimensional assignment problem. *Oper. Res.* 16, 422–431.
The simplest MAP is the 3-DAP. Moreover, most of the results obtained for the 3-DAP can be naturally extended to the MAP, too.

O. Leue (1972). Methoden zur Lösung 3-dimensionaler Zuordnugsprobleme. *Angewandte Mathematik* 14, 154-162.
Two versions of the 3-DAP are stated: the axial 3-DAP and the planar 3-DAP. The feasible solutions of the axial 3-DAP are pair of permutations of $\{1, 2 \ldots, n\}$, whereas there is a one-to-one relation between latin squares and the feasible solutions of the planar 3-DAP.

A.M Frieze (1974). A bilinear programming formulation of the 3-dimensional assignment problems. *Math. Program.* 7, 376–379.
The author considers a slightly generalized form of the axial 3-DAP and gives an equivalent formulation of this problem as a bilinear program. This formulation is exploited then to derive a necessary optimality condition for the axial 3-DAP.

While the axial 3-DAP is NP-hard in the strong sense by a standard observation, the NP-hardness of the planar 3-DAP is shown in the following paper:

A.M. Frieze (1983). Complexity of a 3-dimensional assignment problem. *European J. Oper. Res.* 13, 161–164.

V.A. Jemelichev, M.M. Kovaliev, M.K. Kravtsov (1984). *Polytopes, graphs and optimisation*, Cambridge Univ. Press.

Multidimensional assignment problems are closely related to the multi-index transportation problems. This book provides a detailed analysis of the multi-index transportation problem concerning in particular its polyhedral structure.

13 The Facial Structure of the 3-DAP

The facial structure of the MAP plays an important role in deriving efficient algorithms of branch and cut type. The facial structure of the axial 3-DAP has been investigated in the following three papers. The authors identify facet defining equalities for the corresponding polytope and derive also separation algorithms for these facets.

E. Balas, M.J. Saltzman (1989). Facets of the three-index assignment polytope. *Discr. Appl. Math.* 23, 201–229.
E. Balas, L. Qi (1993). Linear-time separation algorithms for the three-index assignment polytope. *Discr. Appl. Math.* 43 1–12.
L. Qi, E. Balas, G. Gwan (1994). A new facet class and a polyhedral method for the three-index assignment problem. D.-Z. Du (ed.). *Advances in Optimization*, Kluwer Academic, 256–274.

The facial structure of planar 3-DAP has been investigated in the following two papers.

R.E. Burkard, R. Euler, Grommes (1986). On latin squares and the facial structure of related polytopes. *Discr. Math.* 62, 155-181.
R. Euler (1987). Odd cycles and a class of facets of the axial 3-index assignment polytope. *Applicationes Mathematicae (Zastosowania Matematyki)* XIX, 375–386.

14 Algorithms for the 3-DAP

P. Hansen, L. Kaufman (1973). A primal-dual algorithm for the three dimensional assignment problem. *Cahiers Centre Études Rech. Oper.* 15, 327–336.

The authors apply a modified version of the Hungarian method for the linear (two dimensional) assignment problem to the axial 3-DAP.

R.E. Burkard, K. Fröhlich (1980). Some remarks on three dimensional assignment problems. *Methods Oper. Res.* 36, 31–36.

An exact solution method for the axial 3-DAP is derived. This method combines reduction steps with lower bounds computation by subgradient optimization within a branch and bound scheme.

A.M. Frieze, J. Yadegar (1981). An algorithm for solving 3-dimensional assignment problems with applications to scheduling a teaching practice. *Oper. Res.* 32, 989–995.

The authors propose a subgradient optimization method for solving a Lagrangean relaxation of a slightly generalized maximization version of the axial 3-DAP. This algorithm produces quite good solutions on test instances with real life and random input data.

E. Balas, M.J. Saltzman (1991). An algorithm for the three-index assignment problem. *Oper. Res.* 39, 150–161.

A branch and bound algorithm for the axial 3-DAP is derived. The computation of lower bound involves subgradient techniques for solving a Lagrangean relaxation of the problem which incorporates a class of facets defining inequalities. A novel branching strategy exploits the structure of the 3-DAP to reduce the size of the enumeration tree.

R.E. Burkard, R. Rudolf (1993). Computational investigations on 3-dimensional axial assignment problems. *Belgian J. Oper. Res. Stat. Comp. Sci.* 32, 85–98.

This computational study compares different branching rules and bounding procedures for the axial 3-DAP.

A. Poore (1994). Partitioning multiple data sets: Multidimensional assignment and Lagrangean relaxation. *DIMACS Series Discr. Math. Theor. Comp. Sci.* 16, 317–342.

A Lagrangean relaxation method for a class of MAPs is proposed. The relaxed problem is again a MAP and its maximization involves nonsmooth optimization techniques. The algorithm is illustrated and tested on instances arising as mathematical modes of real life data association problems.

A. Poore, A. Robertson (1995). A new Lagrangean relaxation based algorithm for a class of multidimensional assignment problems. *Computational Opt. Appl.* (to appear)

A new Lagrangean relaxation method for sparse MAPs is proposed. The relaxed problem is a linear (two dimensional) assignment problem, whereas the computation of the Lagrangean multipliers involves non-smooth optimization methods.

An interesting application of the MAP arises along with data association in multitarget tracking as described in the following two papers.

A. Poore (1994). Multidimesional assignment formulation of data association problems arising from multitarget and multisensor tracking. *Computational Opt. Appl.* 3, 27–57.
A. Poore (1995). Multidimensional assignment and multitarget tracking. *DIMACS Series Discr. Math. Theor. Comp. Sci.* 19, 169–196.

Compared to the axial 3-DAP less work has been done on the planar 3-DAP. The three following papers describe branch and bound and heuristic methods for this problem.

M. Vlach (1967). A branch and bound method for the three index assignment problem. *Ekonomicko-Matematicky Obzor* 3, 181–191.

A straightforward branch and bound method for the planar 3-DAP is described.

D. Magos, P. Miliotis (1994). An algorithm for the planar three-index assignment problem. *European J. Oper. Res.* 77, 141–153.

A branch and bound algorithm for the planar 3-DAP is proposed and tested. It involves polytomic branching and an improvement method for the computation of the upper bounds. The computation of lower bound is based on Lagrangean relaxations solved by subgradient methods.

D. Magos (1996). Tabu search for the planar three-index assignment problem. *J. Global Opt.* 8, 35–48.

A tabu search algorithm for the planar 3-DAP is proposed and tested on problems of size 5 to 14. The algorithm combines standard tabu search elements such as fixed (variable) tabu list size and frequency-based memory with a new neighborhood structure in the set of latin squares.

15 Polynomially Solvable Cases of the 3-DAP

Polynomially solvable special cases of the axial 3-DAP were singled out in the following two papers.

R.E. Burkard, R. Rudolf, G.J. Woeginger (1996). Three-dimensional axial assignment problems with decomposable cost-coefficients. *Discr. Appl. Math.* 65, 123-139.

This paper investigates 3-DAPs of the form

$$\min_{\phi,\psi\in\mathcal{S}_n} \sum_{i=1}^{n} a_i b_{\phi(i)} c_{\psi(i)}$$

where $\mathcal{S}_n$ is the set of permutations of $\{1, 2, \ldots, n\}$, and shows that in general this problem is NP-hard. Additional conditions on the problem coefficients (a_i), (b_i) and (c_i), $1 \le i \le n$, lead, however, to polynomially solvable special cases. Finally, it is shown that the maximization version of the 3-DAP is also polynomially solvable provided that all coefficients are non-negative.

D. Fortin, R. Rudolf (1995). *Weak algebraic Monge arrays.* SFB Report 33, Institute of Mathematics, Graz University of Technology.

The authors generalize Monge properties to multidimensional arrays and give an explicit optimal solution for MAPs on arrays having such properties.

Other types of special cost coefficients for the MAP and the axial 3-DAP are considered in the following two papers, respectively. Though for the considered special cost coefficients the problems remain NP-hard, polynomial approximation schemes can be given.

H.-J. Bandelt, Y. Crama, F.C.R. Spieksma (1991) Approximation algorithms for multidimensional assignment problems with decomposable costs. *Discr. Appl. Math.* 49, 25–50.
Y. Crama, F.C.R. Spieksma (1992). Approximation algorithms for three-dimensional assignment problems with triangle inequalities. *European J. Oper. Res.* 60, 273–279.

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