A Unified Approach to Simple Special Cases of Extremal Permutation Problems *

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Abstract

Extremal permutation problems are combinatorial problems where an objective function has to be optimized over a set of permutations (as e.g. assignment problems and the traveling salesman problem). In this note we propose a general approach to search for special cases of extremal permutation problems where an optimal solution can be obtained in an explicit form. The approach is based on a special factorization of permutations into transpositions and on a related partial order among permutations. The approach is illustrated by several applications to assignment problems with focus on the quadratic assignment problem. For the Koopmans-Beckmann problem, a number of simple special cases is singled out.

Keywords: Combinatorial optimization, assignment problem, quadratic assignment problem, factorization of permutations, partial order.

1 Introduction

Polynomially solvable cases of NP-hard combinatorial optimization problems give us a possibility to better understand the nature of their intractability. Moreover, they can serve as a base for generating test instances and creating new heuristics.

Extremal permutation problems, i.e. optimization problems over a set of permutations, have a wide range of applications in computer science, combinatorial data analysis and operations research [1, 8, 14]. The linear assignment problem (LAP), the traveling salesman problem (TSP) and the quadratic assignment problem (QAP) constitute the classic core of extremal permutation problems. The reader is referred to the book by Lawler, Lenstra, Rinnooy Kan and Shmoys [12] for getting more information on the TSP and to the survey papers by Burkard [2] and by Pardalos, Rendl and Wolkowicz [15] for information on the QAP.

The LAP is well known to be a polynomially solvable problem, while both the TSP and the QAP are NP-hard. In spite of formal equivalence of the two latter problems from the NP-completeness point of view, the QAP is intuitively recognized to be more complicated. A supporting consideration is that the TSP is easily reducible to the QAP, whereas getting a reverse

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reduction in a direct way seems to be a rather difficult problem. Moreover, many NP-hard problems like finding large cliques, large independent sets or minimum weight feedback arc sets in graphs, like the one dimensional placement problem and others are evident special cases of the QAP. That is why one should estimate special cases of the QAP differently than in the case of the TSP. Unlike optimization problems in graphs, the main feature of the QAP is that its input data are arrays of reals. Hence, also its special cases should be singled out depending on some relations among entries of the arrays. In this note, we will follow this motivation and summarize several polynomially solvable special cases that are obtained in terms of linear inequalities among elements of input matrices of the problems.

Most of the approaches to identify simple special cases of extremal permutation problems can be specified by the three components (C1), (C2) and (C3) below. Apparently, the approach was first used by Demidenko [6, 7] for obtaining simple special cases of the TSP. Throughout this paper the set of permutations of $\{1, 2, \dots, n\}$ is denoted by S_n.

(C1) A set $H \subseteq S_n$ of permutations such that the extremal permutation problem can be solved efficiently over the set H.

(C2) A partial order '$\prec$' on S_n such that all minimal elements of '$\prec$' are in H.

(C3) A property of the input data that makes the objective function nondecreasing with respect to the order '$\prec$'.

A classical example for this approach are *pyramidal tours* in the TSP (cf. Chapter 4 in "The Traveling Salesman Problem" [12]): A pyramidal tour with minimum cost can be found in polynomial time by dynamic programming; this is component (C1). Moreover, there exist special conditions on the distance matrix (like the *Monge* condition [4] and the *Demidenko* conditions [6, 12]) that imply the existence of an optimal tour that is pyramidal. The proof of this fact is done by describing a procedure that transforms a non-pyramidal tour t_1 into another tour t_2 that is 'closer to pyramidality' and does not have higher cost. Setting $t_2 \prec t_1$ under these conditions yields component (C2). The Monge (or Demidenko) condition constitutes component (C3).

This note points out that the above described approach can be useful for identifying easily solvable special cases of other extremal permutation problems, too. This idea is illustrated by applying this approach to identify some polynomially solvable cases of the QAP and some specially solvable cases of the LAP. Although all special cases involved in these applications are simple and almost all of them are well known in the literature, we believe that the unified approach can be successfully used for deriving more interesting and less trivial polynomially solvable special cases of extremal permutation problems. This note is merely a first illustrating step in this direction.

The rest of the paper is organized as follows. In Section 2 we apply the general approach in order to distinguish well solvable cases of extremal permutation problems where the identity permutation ι yields the optimum solution. In this case, $H = \{\iota\}$ and component (C1) is trivial. The partial order in component (C2) exploits a special decomposition of permutations into transpositions. Moreover, we consider certain monotone functions on the set of all permutations S_n. Then in Section 3, we apply the results of Section 2 to derive several sufficient conditions for the identity permutation ι to be an optimal solution of the LAP or the QAP. The sufficient conditions are expressed in terms of systems of linear inequalities on the elements of the instance matrices and constitute component (C3) of the approach. Most of these conditions have already been known in the literature for a long time. However, our general approach allows us to present unified proofs for all these results. The note ends with a short conclusion in Section 4.

2 Definitions and Preliminaries

Throughout the paper, we will use ϕ, ψ, π, σ and ρ to denote permutations in S_n. The identity permutation is denoted by ι. We assume that the multiplication $\phi\psi$ of permutation ϕ with permutation ψ is defined by the relation $\phi\psi(i) = \phi(\psi(i))$. For the sake of readability we will sometimes write $\phi \circ \psi$ instead of $\phi\psi$. These two notations have the same meaning and are alternatively used through the rest of the paper. By ϕ^{-1} we denote the inverse permutation of ϕ.

For an $n \times n$ matrix $A = (a_{ij})$ and $\phi \in S_n$, we denote by A_ϕ the permuted matrix $(a_{\phi(i)\phi(j)})$. For two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, we denote by (A, B) the *scalar product* $\sum_{i=1}^n \sum_{j=1}^n a_{ij}b_{ij}$ of A and B.

Lemma 2.1 *Let ϕ be a permutation in S_n, $\phi \neq \iota$. Then ϕ can be represented in a unique way as product of special transpositions*

$$\phi = \prod_{m=1}^{k} (i_m, m) \tag{1}$$

such that $1 \le i_m \le m$, $1 \le k \le n$ and $i_k \neq k$ holds. Note that there may occur pairs with $i_m = m$ (degenerate transpositions) in the representation (1).

Proof. Let us first argue that for every $\phi \in S_n$, $\phi \neq \iota$, there is at least one representation of the form (1). This statement is evident for $n = 2$. Suppose that it is valid for $n - 1$. If $\phi \in S_n$ and $\phi(n) = n$, then ϕ can be considered to be essentially a permutation of S_{n-1} and the statement holds true by our supposition. Otherwise, $\phi(n) = i_n \neq n$ and by an analogous argument, we get that $\phi \circ (i_n, n) = \prod_{m=1}^{k}(i_m, m)$ holds for some $k \le n-1$, $1 \le i_m \le m$, and $i_k \neq k$. By multiplying the latter equality with the transposition (i_n, n), we obtain (1).

Next we argue that for every $\phi \in S_n$, there is at most one representation of the form (1). Suppose that there exists another factorization $\phi = \prod_{m=1}^{\ell}(j_m, m)$ for some $1 \le j_m \le m$, $1 \le \ell \le n$ and $j_\ell \neq \ell$. If $k > \ell$, the first factorization yields $\phi(k) = i_k$ and the second factorization yields $\phi(k) = k$. Hence $k = i_k$, a contradiction, and we derive $k \le \ell$. Analogously, equality $\ell \le k$ can be derived. Thus $\ell = k$. Finally, we observe that $\phi(i_k) = k$ and $\phi(j_k) = k$ implies $i_k = j_k$, and a straightforward inductive argument completes the proof. ∎

According to Lemma 2.1, we may associate with every permutation $\phi \in S_n$, $\phi \neq \iota$, the unique sequence $\Gamma(\phi) = \langle(1, 1), (i_2, 2), \ldots, (i_k, k)\rangle$ of special transpositions stemming from the representation (1). For the identity permutation ι, we define $\Gamma(\iota)$ to be the empty sequence.

Definition 2.2 *If for two permutations $\phi, \psi \in S_n$ the sequence $\Gamma(\psi)$ is a prefix subsequence of $\Gamma(\phi)$, we say that ψ is a* predecessor *of ϕ and we denote this by $\psi \prec \phi$.*

Clearly, the relation '$\prec$' defines a partial order on S_n, and the identity permutation ι is the unique minimum element of $(S_n, \prec)$. Next, we recall that a function $f : S \to \mathbb{R}$ defined on a partially ordered set $(S, \prec)$ is called *monotonically nondecreasing* if for all $s_1, s_2 \in S$, $s_1 \prec s_2$ implies $f(s_1) \le f(s_2)$. For a function $f : S_n \to \mathbb{R}$, denote by $\Delta f(\phi; i, j) = f(\phi \circ (i, j)) - f(\phi)$ the difference that results when multiplying ϕ with transposition (i, j).

Lemma 2.3 *A function $f : S_n \to \mathbb{R}$ is monotonically nondecreasing on S_n with respect to the partial order '$\prec$', if and only if f fulfills $\Delta f(\phi; i_k, k) \geq 0$ for any permutation $\phi = \prod_{m=1}^{k-1}(i_m, m)$ and for any transposition (i_k, k) with $i_k < k$. In this case, the identity permutation ι yields the minimum value of f on S_n.*

Proof. The (only if) part trivially holds and it remains to prove the (if) part. For permutations $\psi \prec \phi$, let $\phi = \prod_{m=1}^{k}(i_m, m)$ and $\psi = \prod_{m=1}^{\ell}(i_m, m)$ be their decompositions with $\ell \leq k$. Consider the sequence of permutations $\phi_j = \prod_{m=1}^{j}(i_m, m)$, for $\ell \leq j \leq k$. Since $\Delta f(\phi_{j-1}; i_j, j) \geq 0$,

$$f(\phi) - f(\psi) \;=\; \sum_{j=\ell+1}^{k} (f(\phi_j) - f(\phi_{j-1})) \;=\; \sum_{j=\ell+1}^{k} \Delta f(\phi_{j-1}; i_j, j) \;\geq\; 0.$$

Thus, $f(\phi) \geq f(\psi)$ for any $\psi \prec \phi$. Since ι is the unique minimum element of $(S_n, \prec)$, f attains its minimum at ι. ∎

Observe that every permutation $\phi \in S_n$ is uniquely determined by the set containing n pairs $\{(1, \phi(1)), \ldots, (n, \phi(n))\}$. Since for any $\psi \in S_n$, the set

$$\{(\psi(1), \phi(\psi(1))), \ldots, (\psi(n), \phi(\psi(n)))\}$$

determines the same permutation ϕ, a function $f : S_n \to \mathbb{R}$ may be regarded as a mapping satisfying the condition

$$f(\phi) \;=\; f((1, \phi(1)), \ldots, (n, \phi(n))) \;=\; f((\psi(1), \phi\psi(1)), \ldots, (\psi(n), \phi\psi(n)))$$

for arbitrary $\phi, \psi \in S_n$.

Definition 2.4 *Let $\sigma, \rho \in S_n$. A function $g : S_n \to \mathbb{R}$ is said to be σ, ρ-*similar *to a function $f : S_n \to \mathbb{R}$ if*

$$g(\phi) \;=\; g((1, \phi(1)), \ldots, (n, \phi(n))) \;=\; f((\sigma(1), \rho\phi(1)), \ldots, (\sigma(n), \rho\phi(n)))$$

holds for all $\phi \in S_n$.

Observation 2.5 *Let $f : S_n \to \mathbb{R}$ be a monotonically nondecreasing function and let $g : S_n \to \mathbb{R}$ be σ, ρ-similar to f. Then the permutation $\pi = \rho^{-1}\sigma$ yields the minimum value of g on S_n.*

Proof. From Definition 2.4 we easily derive that

$$\begin{aligned} g(\phi) &= g((1, \phi(1)), \ldots, (n, \phi(n))) \;=\; f((\sigma\sigma^{-1}(1), \rho\phi\sigma^{-1}(1)), \ldots, (\sigma\sigma^{-1}(n), \rho\phi\sigma^{-1}(n))) \\ &= f((1, \rho\phi\sigma^{-1}(1)), \ldots, (n, \rho\phi\sigma^{-1}(n))) \;=\; f(\rho\phi\sigma^{-1}). \end{aligned}$$

Hence,

$$g(\pi) \;=\; f(\rho\pi\sigma^{-1}) \;=\; f(\rho\rho^{-1}\sigma\sigma^{-1}) \;=\; f(\iota) \;\leq\; f(\rho\phi\sigma^{-1}) \;=\; g(\phi),$$

where the inequality follows from the monotonicity of f. ∎

Our next goal is to present two sufficient conditions for a function to attain its minimum at the identity permutation ι. Let $U : \mathbb{R}^k \to \mathbb{R}$ and $V : \mathbb{R}^n \to \mathbb{R}$ be real functions, where V is symmetric, i.e. $V(x_1, \ldots, x_n) = V(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ holds for any $\sigma \in S_n$. Furthermore, let

$$t^{(i)},\ i = 1, \ldots, k \qquad \text{and} \qquad h^{(i,j)},\ i, j = 1, \ldots, n$$

be two collections of functions mapping $S_n \to \mathbb{R}$. Now let us define a function $f_U : S_n \to \mathbb{R}$ by

$$f_U(\phi) = U(t^{(1)}(\phi), \ldots, t^{(k)}(\phi))$$

and a function $F_V(\phi, \psi) : S_n \times S_n \to \mathbb{R}$ by

$$F_V(\phi, \psi) = V(h^{(1,\phi(1))}(\psi), \ldots, h^{(n,\phi(n))}(\psi)).$$

Note that for fixed ψ, $F_V(\phi, \psi)$ defines a real function on S_n which we denote by $F_{V,\psi}$. Finally, the restriction of $F_V(\phi, \psi)$ onto $diag(S_n \times S_n)$ yields another real function on S_n which will be denoted by $f_V(\phi)$.

Lemma 2.6 *(a) If $U(x_1, \ldots, x_k)$ is a monotonically nondecreasing function $\mathbb{R}^k \to \mathbb{R}$ with respect to each variable, and if all functions $t^{(i)}$, $i = 1, \ldots, k$, are monotonically nondecreasing on $(S_n, \prec)$, then the identity permutation ι yields the minimum value of $f_U(\phi)$ on S_n.*

(b) If $V(x_1, \ldots, x_n)$ is a symmetric, monotonically nondecreasing function $\mathbb{R}^n \to \mathbb{R}$ with respect to each variable, if all functions $h^{(i,j)}$, $i, j = 1, \ldots, n$, are monotonically nondecreasing on $(S_n, \prec)$, and if ι minimizes $F_{V,\iota}(\phi)$ on S_n, then the identity permutation ι yields the minimum value of $f_V(\phi)$ on S_n.

Proof. Proof of (a). Let ϕ_i yield the minimum of $t^{(i)}$ on S_n. The monotonicity of U on R^k implies that

$$U(t^{(1)}(\phi_1), \ldots, t^{(k)}(\phi_k)) \ \le \ U(t^{(1)}(\phi), \ldots, t^{(k)}(\phi)) \ = \ f_U(\phi)$$

holds for any $\phi \in S_n$. Since $t^{(i)}$ is nondecreasing on $(S_n, \prec)$, by Lemma 2.3 we may take $\phi_i = \iota$, $i = 1, \ldots, k$. Then the righthand side in the above inequality equals $f_U(\iota)$, and the claim follows.

Proof of (b). Let $\psi_{i,j}$ minimize function $h^{i,j}$, $i, j = 1, \ldots, n$ on S_n. Then for $\phi \in S_n$ we have

$$f_V(\phi) \ = \ V(h^{(1,\phi(1))}(\phi), \ldots, h^{(n,\phi(n))}(\phi)) \ \ge \ V(h^{(1,\phi(1))}(\psi_{1,\phi(1)}), \ldots, h^{(n,\phi(n))}(\psi_{n,\phi(n)})).$$

It follows from the conditions of the lemma that

$$V\left(h^{(1,\phi(1))}(\psi_{1,\phi(1)}), \ldots, h^{(n,\phi(n))}(\psi_{n,\phi(n)})\right) \ = \ V\left(h^{(1,\phi(1))}(\iota), \ldots, h^{(n,\phi(1))}(\iota)\right)$$

$$= \ F_{V,\iota}(\phi) \ge F_{V,\iota}(\iota) = V(h^{(1,1)}(\iota), \ldots, h^{(n,n)}(\iota)) \ = \ f_V(\iota).$$

Taking into consideration the previous inequality, we have $f_V(\phi) \ge f_V(\iota)$. ■

3 Well Solvable Cases from Inequalities on Matrix Elements

In this section, we summarize several sufficient conditions for the identity permutation ι to be an optimal solution of the LAP or the QAP. Most of these conditions have already been known in the literature for a long time. We present unified proofs for these conditions that are all based on the results of the preceding section. The sufficient conditions are expressed in terms of systems of linear inequalities on the elements of the instance matrices. All these conditions can be extended and generalized by means of the σ, ρ-similarity introduced in Definition 2.4 and in Observation 2.5 above.

Let us start our investigations with some short remarks on the *Linear Assignment Problem* (LAP). In the LAP, the goal is to minimize for a given $n \times n$ matrix $A = (a_{ij})$ the objective function $v(A, \phi) = \sum_{i=1}^{n} a_{i\phi(i)}$ over all $\phi \in S_n$.

Theorem 3.1 *(see e.g. Rubinstein [17])*
If for an $n \times n$ matrix $A = (a_{ij})$, the matrix elements fulfill the inequalities

$$a_{ik} + a_{kj} - a_{ij} - a_{kk} \;\geq\; 0 \tag{2}$$

for all $1 \leq i, j < k \leq n$, then $v(A, \phi)$ is nondecreasing on $(S_n, \prec)$ and ι yields the minimum value of $v(A, \phi)$ on S_n.

Proof. Let $\prod_{m=1}^{k-1}(i_m, m)$ be the special representation (1) of some permutation $\phi \in S_n$. For $f(\phi) = v(A, \phi)$ and for an arbitrary transposition (i, k) with $i < k$, we have

$$\Delta f(\phi; i, k) \;=\; a_{ik} + a_{k\phi(i)} - a_{i\phi(i)} - a_{kk} \;\geq\; 0,$$

since $1 \leq i, \phi(i) < k$. Now the claim follows from Lemma 2.3. ∎

The matrices that fulfill the inequalities (2) form a superset of the well known class of *Monge* matrices. They are sometimes called *weak Monge* matrices (cf. the survey paper by Burkard, Klinz and Rudolf [4]).

Corollary 3.2 *Let $e_1 \leq e_2 \leq \cdots \leq e_n$ and $d_1 \geq d_2 \geq \cdots \geq d_n$, be real numbers and let matrix $A = (a_{ij})$ be defined by $a_{ij} = e_i d_j$. Then $v(A, \phi)$ is nondecreasing and attains its minimum value at $\phi = \iota$.*

Proof. Since $a_{ik} + a_{kj} - a_{ij} - a_{kk} = (e_i - e_k)(d_k - d_j) \geq 0$ for $1 \leq i, j < k \leq n$, matrix A fulfills the conditions of Theorem 3.1. ∎

Next, we turn to the *General Quadratic Assignment Problem* (GQAP) with a 4-index array of coefficients $C = (c_{ijkl})$, $1 \leq i, j, k, l \leq n$. The GQAP consists in minimizing the objective function

$$w(C, \phi) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij\phi(i)\phi(j)} \tag{3}$$

over all $\phi \in S_n$. We introduce auxiliary matrices

$$\begin{array}{llll} A^{<s,t>} = (a_{ij}^{<s,t>}), & a_{ij}^{<s,t>} = c_{sitj}, & s, t = 1, \ldots, n \\ B^{<s,t>} = (b_{ij}^{<s,t>}), & b_{ij}^{<s,t>} = c_{isjt}, & s, t = 1, \ldots, n \end{array} \tag{4}$$

For a triple of indices i, j, k with $1 \le i, j < k \le n$ we set

$$\Delta_{ijk}(A^{<s,t>}) = a_{ik}^{<s,t>} + a_{kj}^{<s,t>} - a_{ij}^{<s,t>} - a_{kk}^{<s,t>}. \tag{5}$$

Note that for a permutation $\phi = \prod_{m=1}^{k-1}(i_m, m)$ in S_n with $\phi(i) = j$, for the transposition (i,k) with $i < k$, and for the function $g(\phi) = v(A^{<s,t>}, \phi)$, the value $\Delta_{ijk}(A^{<s,t>})$ exactly denotes the function change $\Delta g(\phi; i, k)$.

Lemma 3.3 *Let $f(\phi) = w(C, \phi)$ denote the objective function of the GQAP. Then*

$$\begin{aligned}\Delta f(\phi; i,k) \quad &= \quad \sum_{\substack{s=1\\ s\neq i,k}}^{n} \Big(\Delta_{ijk}(A^{<s,\phi(s)>}) + \Delta_{ijk}(B^{<s,\phi(s)>})\Big) + \\ &\quad + \Delta_{ijk}(A^{<i,k>}) + \Delta_{ijk}(A^{<k,j>}) - \Delta_{ijk}(A^{<i,j>}) - \Delta_{ijk}(A^{<k,k>})\end{aligned}$$

holds for any permutation $\phi = \prod_{m=1}^{k-1}(i_m, m)$ where $1 \le i < k$ and $1 \le j = \phi(i) < k$.

Proof. Let $\phi = \prod_{m=1}^{k-1}(i_m, m)$, $1 \le k \le n$, be a permutation of S_n. Then

$$\begin{aligned}f(\phi) \quad = \quad & \sum_{\substack{r=1\\ r\neq i, r\neq k}}^{n} \sum_{\substack{s=1\\ s\neq i, s\neq k}}^{n} c_{rs\phi(r)\phi(s)} + \sum_{\substack{r=1\\ r\neq i, r\neq k}}^{n} c_{ri\phi(r)\phi(i)} + \sum_{\substack{r=1\\ r\neq i, r\neq k}}^{n} c_{rk\phi(r)k} + \\ & + \sum_{\substack{r=1\\ r\neq i, r\neq k}}^{n} c_{ir\phi(i)\phi(r)} + \sum_{\substack{r=1\\ r\neq i, r\neq k}}^{n} c_{krk\phi(r)} + c_{ik\phi(i)k} + c_{ii\phi(i)\phi(i)} + c_{kik\phi(i)} + c_{kkkk}.\end{aligned}$$

For a transposition (i,k) with $i < k$ we have

$$\begin{aligned}f(\phi \circ (i,k)) \quad = \quad & \sum_{\substack{r=1\\ r\neq i, r\neq k}}^{n} \sum_{\substack{s=1\\ s\neq i, s\neq k}}^{n} c_{rs\phi(r)\phi(s)} + \sum_{\substack{r=1\\ r\neq i, r\neq k}}^{n} c_{ri\phi(r)k} + \sum_{\substack{r=1\\ r\neq i, r\neq k}}^{n} c_{rk\phi(r)\phi(i)} + \\ & + \sum_{\substack{r=1\\ r\neq i, r\neq k}}^{n} c_{irk\phi(r)} + \sum_{\substack{r=1\\ r\neq i, r\neq k}}^{n} c_{kr\phi(i)\phi(r)} + c_{iikk} + c_{ikk\phi(i)} + c_{ki\phi(i)k} + c_{kk\phi(i)\phi(i)}.\end{aligned}$$

For $j = \phi(i)$ this yields

$$\begin{aligned}\Delta f(\phi; i,k) \quad &= \quad f(\phi \circ (i,k)) - f(\phi) \\ &= \quad \sum_{\substack{r=1\\ r\neq i, r\neq k}}^{n} \Big(c_{ri\phi(r)k} + c_{rk\phi(r)j} - c_{ri\phi(r)j} - c_{rk\phi(r)k}\Big) + \\ &\quad + \sum_{\substack{r=1\\ r\neq i, r\neq k}}^{n} \Big(c_{rk\phi(r)} + c_{krj\phi(r)} - c_{irj\phi(r)} - c_{krk\phi(r)}\Big) + \\ &\quad + c_{iikk} + c_{ikkj} + c_{kijk} + c_{kkjj} - c_{ikjk} - c_{iijj} - c_{kikj} - c_{kkkk}.\end{aligned}$$

It follows from equations (4) and (5) that

$$\begin{aligned} c_{ri\phi(r)k} + c_{rk\phi(r)j} - c_{ri\phi(r)j} - c_{rk\phi(r)k} &= \Delta_{ijk}(A^{<r,\phi(r)>}) \\ c_{irk\phi(r)} + c_{krj\phi(r)} - c_{irj\phi(r)} - c_{krk\phi(r)} &= \Delta_{ijk}(B^{<r,\phi(r)>}). \end{aligned}$$

Moreover, the equality

$$\begin{aligned} &\Delta_{ijk}(A^{<i,k>}) + \Delta_{ijk}(A^{<k,j>}) - \Delta_{ijk}(A^{<i,j>}) - \Delta_{ijk}(A^{<k,k>}) \\ &\qquad = c_{iikk} + c_{ikkj} + c_{kijk} + c_{kkjj} - c_{ikjk} - c_{iijj} - c_{kikj} - c_{kkkk} \end{aligned}$$

is straightforward to verify. Then, substituting these expressions into the expression for $\Delta f(\phi; i, k)$ completes the proof. ■

Theorem 3.4 *Let $C = (c_{ijkl})$, $1 \le i,j,k,l \le n$, be a matrix whose elements fulfill the inequalities*

$$0 \le c_{sitk} + c_{sktj} - c_{sitj} - c_{sktk} \tag{6}$$

$$0 \le c_{isks} + c_{ksjs} - c_{isjs} - c_{ksks} \tag{7}$$

for all $1 \le i, j < k \le n$ and $1 \le s,t \le n$. Then, the identity permutation ι yields the minimum value of the GQAP objective function $w(C,\phi)$ over all $\phi \in S_n$.

Proof. The objective function $w(C,\phi)$ may be interpreted as a function $f_V(\phi)$ as defined in Section 2. Indeed, by taking into consideration (4), we get that

$$w(C,\phi) = \sum_{r=1}^{n}\sum_{s=1}^{n} c_{rs\phi(r)\phi(s)} = \sum_{r=1}^{n}\left(\sum_{s=1}^{n} a_{s,\phi(s)}^{<r,\phi(r)>}\right) = \sum_{r=1}^{n} v(A^{<r,\phi(r)>},\phi) = f_V(\phi),$$

where $V(x_1,\ldots,x_n) = \sum_{r=1}^{n} x_r$ is a symmetric, monotonically nondecreasing function $\mathbb{R}^n \to \mathbb{R}$ and $h^{(r,s)}(\phi) = v(A^{<r,s>},\phi)$, $r,s = 1,\ldots,n$, are functions $S_n \to \mathbb{R}$. By using (4), the function $F_{V,\psi}(\phi)$ may be written as

$$F_{V,\psi}(\phi) = \sum_{r=1}^{n}\sum_{s=1}^{n} c_{rs\phi(r)\psi(s)} = \sum_{s=1}^{n}\left(\sum_{r=1}^{n} b_{r\phi(r)}^{<s,\psi(s)>}\right) = \sum_{s=1}^{n} v(B^{<s,\psi(s)>},\phi).$$

By setting $\psi = \iota$ in the latter equation, one derives $F_{V,\iota}(\phi) = \sum_{s=1} v(B^{<s,s>},\phi)$. Combining conditions (6) and (7) with the statement of Theorem 3.1 ensures the monotonicity of all functions $v(A^{<r,s>},\phi)$ and $v(B^{<s,s>},\phi)$, $1 \le r,s \le n$, on $(S_n, \prec)$. Hence, Lemma 2.6(a) shows that $\phi = \iota$ yields the minimum of $F_{V,\iota}(\phi)$ on S_n. In turn, Lemma 2.6(b) then yields that $f_V(\phi) = w(C,\phi)$ attains its minimum at $\phi = \iota$. ■

Next, let us turn to the Koopmans-Beckmann [9] version of the quadratic assignment problem where $c_{ijkl} = a_{ij}b_{kl}$, for all $1 \le i,j,k,l \le n$. Here $A = (a_{ij})$ and $B = (b_{ij})$ are two $n \times n$ matrices. In this case, minimizing the objective function $w(C,\phi)$ as defined in (3) is equivalent to minimizing the scalar product

$$(A, B_\phi) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} b_{\phi(i)\phi(j)} \tag{8}$$

over all permutations $\phi \in S_n$.

Theorem 3.5 *(Krushevski 1964, [10])*
If the $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ fulfill the inequalities

$$a_{is} \leq a_{js}, \quad a_{si} \leq a_{sj}; \qquad b_{is} \geq b_{js}, \quad b_{si} \geq b_{sj} \tag{9}$$

for all $1 \leq i < j \leq n$ and $1 \leq s \leq n$, then $(A, B_\iota) \leq (A, B_\phi)$ for all $\phi \in S_n$.

Proof. Define a 4-index array $C = (c_{ijkl})$ by $c_{ijkl} = a_{ij}b_{kl}$, for all $1 \leq i, j, k, l \leq n$. Then for all $1 \leq i, j < k \leq n$ and $1 \leq s \leq n$, the inequalities

$$\begin{aligned} c_{sitk} + c_{sktj} - c_{sitj} - c_{sktk} &= (a_{sk} - a_{si})(b_{tj} - b_{tk}) \geq 0 \\ c_{isks} + c_{ksjs} - c_{isjs} - c_{ksks} &= (a_{ks} - a_{is})(b_{js} - b_{ks}) \geq 0 \end{aligned}$$

hold and C fulfills the conditions of Theorem 3.4. Since $w(C, \phi) = (A, B_\phi)$ for any $\phi \in S_n$, Theorem 3.4 yields that $(A, B_\iota) = w(C, \iota) \leq w(C, \phi) = (A, B_\phi)$ for all $\phi \in S_n$. ■

Theorem 3.6 *If the $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ fulfill the inequalities*

$$a_{11} \leq \ldots \leq a_{nn}; \qquad b_{11} \geq \ldots \geq b_{nn}; \qquad a_{ks} + a_{sk} = a_{is} + a_{si}; \qquad b_{st} = b_{ts} \tag{10}$$

for all $1 \leq i < k \leq n$ and $1 \leq s, t \leq n$, then the function (A, B_ϕ) mapping $S_n \to \mathbb{R}$ is nondecreasing on $(S_n, \prec)$ and $(A, B_\iota) \leq (A, B_\phi)$ holds for all $\phi \in S_n$.

Proof. Define a matrix $C = (c_{ijkl})$ by $c_{ijkl} = a_{ij}b_{kl}$. Then

$$\begin{aligned} &\Delta_{ijk}(A^{<s,\phi(s)>}) + \Delta_{ijk}(B^{<s,\phi(s)>}) = \\ &\quad = a_{si}b_{\phi(s)k} + a_{sk}b_{\phi(s)j} + a_{is}b_{\phi(s)k} + a_{ks}b_{\phi(s)j} - a_{si}b_{\phi(s)j} - a_{sk}b_{\phi(s)k} - a_{is}b_{\phi(s)j} - a_{ks}b_{\phi(s)k} \\ &\quad = (a_{si} + a_{is} - a_{sk} - a_{ks})b_{\phi(s)k} + (a_{sk} + a_{ks} - a_{si} - a_{is})b_{\phi(s)j} \ = \ 0, \end{aligned}$$

and

$$\Delta_{ijk}(A^{<i,k>}) + \Delta_{ijk}(A^{<k,j>}) - \Delta_{ijk}(A^{<i,j>}) - \Delta_{ijk}(A^{<k,k>}) \ = \ (a_{kk} - a_{ii})(b_{jj} - b_{kk}) \ \geq \ 0.$$

Plugging these results into the expression for the function change of $f(\phi) = (A, B_\phi)$ as derived in Lemma 3.3 yields

$$\Delta f(\phi; i, j) \ = \ (a_{kk} - a_{ii})(b_{jj} - b_{kk}) \ \geq \ 0,$$

which together with the statement of Lemma 2.3 completes the proof of the theorem. ■

Theorem 3.7 *(Krushevski 1965, [11])*
Let $e_1 \leq e_2 \leq \cdots \leq e_n$, $d_1 \geq d_2 \geq \cdots \geq d_n$, $E = \sum_{i=1}^n e_i$, $D = \sum_{i=1}^n d_i$, and α_1, α_2, β_1, β_2 be real numbers. Let the two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ be defined by $a_{ij} = \beta_1 d_i + \beta_2 d_j + d_i d_j$ and by $b_{ij} = \alpha_1 e_i + \alpha_2 e_j + e_i e_j$. Moreover, let

$$W = (\alpha_1 + \alpha_2)D + (\beta_1 + \beta_2)E + n(\alpha_1\beta_1 + \alpha_2\beta_2).$$

If $\sum_{i=1}^n e_i d_i \geq -W/2$, then $(A, B_\iota) \leq (A, B_\phi)$ holds for all $\phi \in S_n$.

Proof. Let $M = (m_{ij})$ be defined by $m_{ij} = e_i d_j$ and let $t(\phi) = v(M, \phi)$. By Corollary 3.2, function $t(\phi)$ is nondecreasing on $(S_n, \prec)$. Thus, $t(\phi)$ attains its minimum at $\phi = \iota$ and therefore, the inequality $t(\phi) \geq t(\iota) = \sum_{i=1}^n e_i d_i \geq -W/2$ holds, for all $\phi \in S_n$. Note that

$$\begin{aligned} (A, B_\phi) &= \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{\phi(i)\phi(j)} \\ &= \sum_{i=1}^n \sum_{j=1}^n (\beta_1 d_i + \beta_2 d_j + d_i d_j)(\alpha_1 e_{\phi(i)} + \alpha_2 e_{\phi(j)} + e_{\phi(i)} e_{\phi(j)}) \\ &= t^2(\phi) + W t(\phi) + (\alpha_1 \beta_2 + \alpha_2 \beta_1) ED. \end{aligned}$$

Let $T : \mathbb{R} \to \mathbb{R}$, $T(x) = x^2 + Wx + (\alpha_1 \beta_2 + \alpha_2 \beta_1) ED$ and note that T is nondecreasing for $x \geq -W/2$. Hence, the conditions of Lemma 2.6(a) are fulfilled and the claim follows. ■

We remark that the condition "$\sum_{i=1}^n e_i d_i \geq -W/2$" in the statement of the above theorem is essential: Without this condition, the form (A, B_ϕ) does not necessarily take its minimum for $\phi = \iota$. It can be even shown that finding the minimum in this case is an NP-complete problem.

4 Conclusions and Remarks

We presented a unified proof scheme for deriving simple special cases of the linear assignment problem and the quadratic assignment problem, in which the identity permutation constitutes an optimal solution. By this, we derived unified proofs for ancient results of Krushevski [10, 11] and Rubinstein [17]. Although the special cases treated here are simple, the basic idea of our approach as described in the introduction is rather general and was proven to be useful for identifying very interesting polynomially solvable cases of the TSP (see Chapter 4 of [12]). We believe that the same idea can also be used for deriving more interesting and less trivial polynomially solvable cases of other permutation problems, e.g. the QAP, this note being merely the very first illustrating step in this direction.

Considering that the identical permutation constitutes an optimal solution for all special cases of extremal permutation problems treated in this paper, the following question arises. Is the approach described in this paper a general proof method for all extremal permutation problems whose solution can be explicitly given independently from the coefficients of the considered problem instance? The answer of this question is negative. In [3] we have investigated some special cases of the QAP which cannot be handled by using the unified approach described in this paper, although in all those cases an optimal solution can be explicitly given as described above. In particular, the known special cases of the one-dimensional module placement problem as investigated by Burkov, Rubinstein and Sokolov [5], by Metelski [13] and by Pratt [16]) remain out of reach of our approach.

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