Risk theory and risk management in actuarial science
Winter term 2016/17
4th work sheet

20. Let the random variables X_i, $i = 1, 2$, be such that $X_1 \sim Exp(\lambda)$ and $X_2 = X_1$, where $Exp(\lambda)$ is the exponential distribution with parameter λ. Consider the strictly increasing functions $t_i\colon \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, with $t_1(x) = x$ and $t_2(x) = x^2$. Show that the following equalities for the linear correlation coefficient ρ_L hold:
$$\rho_L(X_1, X_2) = 1 \text{ and } \rho_L\Big(t_1(X_1), t_2(X_2)\Big) = \frac{2}{\sqrt{5}}\,.$$

21. Let the random variables X_i, $i = 1, 2$, be such that $X_1 \sim Exp(\lambda)$ and $X_2 = X_1^2$, where $Exp(\lambda)$ is the exponential distribution with parameter λ. Determine the coefficients of the lower and the upper tail dependence $\lambda_L(X_1, X_2)$, $\lambda_U(X_1, X_2)$, respectively, and conclude that X_1 and X_2 have both a lower and an upper tail dependence. Compute also the coefficient of the linear correlation $\rho_L(X_1, X_2)$, compare the three computed dependence measures and comment on your results.

22. Show that VaR_α, $\alpha \in (0, 1)$, is not a coherence risk measure (in general). To this end you can analyze the properties of $VaR_\alpha(X)$ for a binomially distributed random variable X with $X \sim B(p, n)$, where $B(p, n)$ is a binomial distribution with parameters $p \in (0, 1)$ and $n \in \mathbb{N}$.

23. Let $h\colon \mathbb{R} \to \mathbb{R}$ be a monotone increasing function with $h(\mathbb{R}) = \mathbb{R}$ and let $h^{\leftarrow}\colon \mathbb{R} \to \mathbb{R}$ be generalized inverse function of h. Show that the following statements hold.
    (a) h is continuous if and only if $h^{\leftarrow}$ is strictly monotone increasing .
    (b) h is strictly monotone increasing if and only if $h^{\leftarrow}$ is continuous.
    (c) $h^{\leftarrow}(h(x)) \le x$
    (d) If h is strictly monotone increasing then $h^{\leftarrow}(h(x)) = x$.
    (e) h is continuous if and only if $h(h^{\leftarrow}(y)) = y$.

24. Let X be a random variable with distribution function F. Show that $F^{\leftarrow}(F(X))$ is almost surely equal to X, i.e. the equality $Prob\,(F^{\leftarrow}(F(X)) = X) = 1$ holds.

25. Show that the Fréchet lower bound W_d is not a copula for $d \ge 3$.

    Hint: Show that the rectangle inequality
$$\sum_{k_1=1}^{2}\sum_{k_2=1}^{2}\cdots\sum_{k_d=1}^{2}(-1)^{k_1+k_2+\ldots+k_d}W_d(u_{1k_1}, u_{2k_2}, \ldots, u_{dk_d}) \ge 0\,,$$
    where $(a_1, a_2, \ldots, a_d)$, $(b_1, b_2, \ldots, b_d) \in [0, 1]^d$ with $a_k \le b_k$ and $u_{k1} = a_k$ und $u_{k2} = b_k$ for all $k \in \{1, 2, \ldots, d\}$, is violated if $d \ge 3$ and $a_i = \frac{1}{2}$, $b_i = 1$, for $i = 1, 2, \ldots, d$.

26. Let X_i, $i = 1, 2$, be two lognormally distributed random variables with $X_1 \sim Lognormal(0, 1)$ und $X_2 \sim Lognormal(0, \sigma^2)$, $\sigma > 0$. Compute $\rho_{L,min}(X_1, X_2)$ und $\rho_{L,max}(X_1, X_2)$ in dependence of σ and compare their values for different values of $\sigma > 0$.

27. Construct two random vectors $(X_1, X_2)^T$ and $(Y_1, Y_2)^T$ with different joint distributions $F_{(X_1,X_2)}$, $F_{(Y_1,Y_2)}$, respectively, such that (a) all X_1, X_2, Y_1, Y_2 are standard normally distributed, i.e. $X_1, X_2, Y_1, Y_2 \sim N(0, 1)$, (b) the two X-variables and the two Y-variables are uncorrelated, i.e. $\rho_L(X_1, X_2) = 0$, $\rho_L(Y_1, Y_2) = 0$, and (c) the α-quantiles of the corresponding sums are different, i.e. $F^{\leftarrow}_{X_1+X_2}(\alpha) \ne F^{\leftarrow}_{Y_1+Y_2}(\alpha)$ holds for some $\alpha \in (0, 1)$, where $F_{X_1+X_2}$, $F_{Y_1+Y_2}$ are the distributions of $X_1 + X_2$ and $Y_1 + Y_2$, respectively.

    Conclude that in general it is not possible to draw conclusions about the loss of a portfolio if the loss distributions of the single assets in portfolio and their mutual linear correlation coefficients are known.