Operations Research
Winter term 2015/2016
7th work sheet (multicriteria optimisation)

36. Let $X = \{(x_1, x_2) \in \mathbb{R}^2 \colon \mathrm{x}_1 \geq 0, \mathrm{x}_2 \geq 0, \mathrm{x}_1 + \mathrm{x}_2 \leq 100, 2\mathrm{x}_1 + \mathrm{x}_2 \leq 150\}$, $f_1(x_1, x_2) = -6x_1 - 4x_2$ and $f_2(x_1, x_2) = -x_1$. Solve the *ϵ-constrained problem* $P_1(\epsilon)$ for $\epsilon = 0$ cf. lecture for the definition of ϵ-constrained problems). Use the method of Benson to check whether the optimal solution x^* of $P_1(0)$ is Pareto-optimal to $(X, f, \mathbb{R}^2)/\mathrm{id}/(\mathbb{R}^2, <)$ or not.

37. Consider $X = \{(x_1, x_2) \in \mathbb{R}^2 \colon (\mathrm{x}_1 - 1)^2 + (\mathrm{x}_2 - 1)^2 \leq 1\} + \mathbb{R}^2_+$ and $f \colon X \to \mathbb{R}^2$ with $f(x) = x$, $\forall x \in X$. Let $(\epsilon_1, \epsilon_2) \in \mathbb{R}^2_+$ with $\epsilon_1 > 1$ and $(\mu_1, \mu_2) \in \mathbb{R}^2_+$ with $\mu_1 > 0$. Solve the problem $P_2(\epsilon, \mu)$ (cf. lecture). Show that its optimal solution is a weakly Pareto optimal solution but not a Pareto optimal solution to $(X, f, \mathbb{R}^2)/\mathrm{id}/(\mathbb{R}^2, \leq)$, and that this holds for all $\epsilon_1 > 1$ and for all $\mu_1 > 0$.

38. Consider $X = \{(x_1, x_2) \in \mathbb{R}^2 \colon (\mathrm{x}_1 - 1)^2 + (\mathrm{x}_2 - 1)^2 \leq 1\}$ and $f \colon X \to \mathbb{R}^2$ with $f(x) = x$, $\forall x \in X$. Solve the problem $P_2(\epsilon, \mu)$ (cf. lecture) with $\epsilon = 0$ in order to illustrate that there exist (non-proper) Pareto optimal solutions to $(X, f, \mathbb{R}^2)/\mathrm{id}/(\mathbb{R}^2, \leq)$ which can not be obtained as optimal solutions of $P_2(\epsilon, \mu)$ for some $0 \leq \mu < \infty$. Could this Pareto optimal solution of $(X, f, \mathbb{R}^2)/\mathrm{id}/(\mathbb{R}^2, \leq)$ be obtained as an optimal solution of $P_2(\epsilon, \mu)$ with $0 \leq \mu < \infty$ for some other value of $\epsilon \neq 0$?

39. Consider $(X, f, \mathbb{R}^{\mathrm{n}})/\mathrm{id}/(\mathbb{R}^{\mathrm{Q}}, \leq)$ and assume $0 < \min_{x \in X} f_k(x)$, for all $k = 1, 2, \ldots, Q$. Prove that $x \in X_{\mathrm{wPar}}$ if and only if x is an optimal solution of $\min_{x \in X} \max_{k=1,2,\ldots,Q} \lambda_k f_k(x)$, for some $\lambda \in \mathrm{int}(\mathbb{R}^{\mathrm{Q}}_+)$.

40. Consider finding a compromise solution by maximizing the distance to the nadir point. Let $|| \cdot ||$ be a norm. Show that an optimal solution of the problem $\max\{||f(x) - y^N|| \colon x \in X\}$ is weakly Pareto optimal to $(X, f, \mathbb{R}^{\mathrm{n}})/\mathrm{id}/(\mathbb{R}^{\mathrm{Q}}, \leq)$. Give a condition under which an optimal solution of the above maximization problem is guaranted to be a Pareto optimal solution of $(X, f, \mathbb{R}^{\mathrm{n}})/\mathrm{id}/(\mathbb{R}^{\mathrm{Q}}, \leq)$.

41. Solve the problem in example 36 by means of the compromise programming approach. Use $\lambda = (1/2, 1/2)$ and identify the solution of CP^w_p für $p = 1, 2, \infty$.

42. Let $Y = \{y = (y_1, y_2) \in \mathbb{R}^2_+ \colon \mathrm{y}_1^2 + \mathrm{y}_2^2 \geq 1\}$. Show the existence of a parameter p, $1 < p < \infty$, such that the following equality holds:
$$Y_{\mathrm{eff}} = \cup_{w \in \Lambda^0} A(\lambda, p, Y).$$
Use the ideal point y^I or the utopic point y^U in the definition of $A(w, p, Y)$ and N_p^λ.

43. Consider $(X, f, \mathbb{R}^{\mathrm{n}})/\mathrm{id}/(\mathbb{R}^{\mathrm{Q}}, \leq)$ and assume that $\hat{x} \in X_{\text{p-Par}}$. Show that there exist $\epsilon \in \mathbb{R}^{\mathrm{Q}}$, $\mu \in \mathbb{R}^{\mathrm{Q}}_+$ and $\hat{s} \in \mathbb{R}^{\mathrm{Q}}_+$, such that $(\hat{x}, \hat{s})$ is an optimal solution to $P_j(\epsilon, \mu)$, $\forall j \in \{1, 2, \ldots, Q\}$. Thus *proper* Pareto optimal solution can be obtained as optimal solutions of the elastic constrained problem $P_j(\epsilon, \mu)$ with finite penalties μ_j, $j \in \{1, 2, \ldots, Q\}$.