Operations Research
Winter term 2015/2016
6th work sheet (multicriteria optimisation)

31. (Ehrgott et al., 1997[1])
Let $X \subseteq \mathbb{R}^n$, $f\colon X \to \mathbb{R}$, and $\bar{x} \in X$. The set $L_{\leq}(f(\bar{x})) = \{x \in X\colon f(x) \leq f(\bar{x})\}$ is called a *level set* of f in $\bar{x}$, the set $L_{=}(f(\bar{x})) = \{x \in X\colon f(x) = f(\bar{x})\}$ is called *level curve* of f in $\bar{x}$ and $L_{<}(f(\bar{x})) = \{x \in X\colon f(x) < f(\bar{x})\}$ is called *strict level set* of f in $\bar{x}$. Consider an MCOP $(X, f, \mathbb{R}^Q)/\mathrm{id}/(\mathbb{R}^Q, <)$. Let $x^* \in X$, $f = (f_1, f_2, \ldots, f_q)^T$ and $y_q := f_q(x^*)$ for $q = 1, 2, \ldots, Q$. Show that
   (a) x^* is strict Pareto optimal if and only if $\cap_{q=1}^{Q} L_{\leq}(y_q) = \{x^*\}$, where $L_{\leq}(y_q)$ is the level set of f_q in x^*, $q = 1, 2, \ldots, Q$.
   (b) x^* is Pareto optimal if and only if $\cap_{q=1}^{Q} L_{\leq}(y_q) = \cap_{q=1}^{Q} L_{=}(y_q)$, where $L_{\leq}(y_q)$ is as in (a) and $L_{=}(y_q)$ is the level curve of f_q in y_q, $q = 1, 2, \ldots, Q$.
   (c) x^* ist weakly Pareto optimal if and only if $\cap_{q=1}^{Q} L_{<}(y_q) = \emptyset$, where $L_{<}(y_q)$ is the strict level set of f_q in y_q, $q = 1, 2, \ldots, Q$.

32. Let $[a, b] \in \mathbb{R}$ be a compact interval and the functions $f_i\colon \mathbb{R} \to \mathbb{R}$ convex, $i = 1, 2, \ldots, Q$. Let
$$x_i^m := \min\{x \in [a, b]\colon f_i(x) = \inf_{x \in [a,b]} f_i(x)\} \text{ and}$$
$$x_i^M := \max\{x \in [a, b]\colon f_i(x) = \inf_{x \in [a,b]} f_i(x)\}$$
Use the result of Exercise 31 to show the following identities:
$$X_{Par} = \left[\min_{i=1,2,\ldots,Q} x_i^M, \max_{i=1,2,\ldots,Q} x_i^m\right] \bigcup \left[\max_{i=1,2,\ldots,Q} x_i^m, \min_{i=1,2,\ldots,Q} x_i^M\right]$$
$$X_{w-Par} = \left[\min_{i=1,2,\ldots,Q} x_i^m, \max_{i=1,2,\ldots,Q} x_i^M\right]$$

33. Use the results of Exercise 32 to give an example of a multicriteria optimisation problem with $X \subset \mathbb{R}$, where $X_{s-Par} \subsetneq X_{Par} \subsetneq X_{wPar}$, with strict inclusions. Use two or three objective functions.

34. Let $X = \{x \in \mathbb{R}\colon \mathrm{x} \geq 0\}$ and $f_1(x) = e^x$,
$$f_2(s) = \begin{cases} \frac{1}{x+1} & 0 \leq x \leq 5 \\ (x-5)^2 + \frac{1}{6} & x \geq 5\,. \end{cases}$$
Using the results of Exercise 32, determine X_{Par}. Which of these solutions are strictly Pareto? Can you prove a sufficient condition on f for $x \in \mathbb{R}$ to be a strict Pareto optimal solution of $(X, f, R^Q)/id/(R^Q, \leq)$, where $X \subset \mathbb{R}$ and $f\colon X \to R^Q$, $x \mapsto \big(f_i(x)\big)_{1 \leq i \leq Q}$.

35. Provide an example in which the following relationships hold, respectively:
   (a) $S(Y) \subset Y_{eff} \subset S_0(Y)$ where both inclusions are strict and $S(Y)$, $S_0(Y)$ are defined as in the lecture.
   (b) $S(Y) \cup S_0'(Y) = Y_{eff} = S_0(Y)$, where
$$S_0'(Y) = \left\{ y' \in Y\colon \exists \lambda \in \mathbb{R}_+^Q \setminus \{0\} \text{ such that } \{y'\} = S(\lambda, Y) \right\}.$$
$S(\lambda, Y)$, $S(Y)$ and $S_o(Y)$ are defined as in the lecture, namely
$$S(\lambda, Y) = \operatorname{argmin}\{\langle \lambda, y\rangle\colon y \in Y\}\,,\ S(Y) = \cup_{\lambda \in Int(\mathbb{R}_+^Q)} S(\lambda, Y)\,,\ \text{and } S_0(Y) = \cup_{\lambda \in \mathbb{R}_+^Q \setminus \{0\}} S(\lambda, Y)\,.$$

[1] M. Ehrgott, Multicriteria Optimization, Second Edition, Springer, Berlin-Heidelberg-New York, 2005