Operations Research
Winter term 2015/2016

1st work sheet (dynamic optimization)

1. Consider the deterministic dynamic programming problem $(N, \mathcal{S}, \mathcal{A}, r_n, z_n, V_N)$ with a finite planing horizon of length N as introduced in the lecture. Let the set $\mathcal{S}$ of states be countable and non-empty and the set of actions $\mathcal{A}$ be finite and non-empty. Denote by $S_n \subseteq \mathcal{S}$ the set of possible states in period n, by $A_n(s_n)$ the set of possible actions taken in state s_n at period n, and by $r_n(s_n, a_n)$ the one step gain function in period n at state s_n if the action taken was $a_n \in A_n(s_n)$. Further, let z_n be the state transformation function in period n such that $s_{n+1} := z_n(s_n, a_n) \in S_{n+1}$ is the state which results after taking action $a_n \in A_n(s_n)$ in state $s_n \in S_n$. Finally, $V_N(s_N)$ is the terminal gain function which depends on the state s_N at the last period N. Recall that in the model discussed in the lecture the state s_{n+1} in period $n+1$ depends just on the state s_n in period n and the action a_n taken at that state. For this reason the problem is called a *first order dynamic programming problem*. Analogously in a *second order dynamic programming problem* the state s_{n+1} in period $n+1$ depends on the state s_n in period n and the action a_n taken at that state, and additionally also on state s_{n-1} in period $n-1$.

   Show how to transform a second order dynamic programming problem into a first order dynamic programming problem by modifying the state variables.

2. Let the demand r_i on a certain component be given as follows for the next five periods i, $1 \leq i \leq 5$: $r_1 = 2$, $r_2 = 4$, $r_3 = 2$, $r_4 = 2$ und $r_5 = 4$. The components have to be produced and the production involves setup costs of 5 Euro, production costs of 1 Euro per piece, and inventory costs of $0,30$ Euro per piece and month. Assume that the whole production of a month happens at the beginning of the month and that in every month the whole demand is delivered to the customer immediately after the production at the beginning of the month. Determine by means of dynamic programming a production plan which fulfills the demand and minimizes the total costs.

3. A taxi company consumes 8500 liter fuel per month. The consumption happens continuously at a constant rate. The costs of the fuel amount to 1.20 Euro per liter, the fixed ordering costs and the inventory costs amount to 1000 euro per order and 1 Cent per liter and month, respectively. Determine an ordering plan (i.e. when to order and which quantity to order, respectively) which minimizes the overall costs and avoids shortfalls.

4. Six crew members of a sales division will be assigned to three regions. Every region should become at least one crew meber to supervise it and every crew member should supervise just one region. Table 4 shows an estimation (in a suitable unit) for the increase of the business volume in each region depending on the number of crew members assigned to it. Determine how many crew members should be assigned to each reagion such that the overall increase of the business volume is maximized. Solve this problem by means of dynamic programming.

Number of crew members	Region 1	Region 2	Region 3
1	4	3	5
2	6	6	7
3	9	8	10
4	11	10	12

Table 1: Data for problem 4

5. Three teams of scientists are dealing with a difficult, yet unsolved, problem. The failure probability of the teams, denoted by i, $1 \leq i \leq 3$, is estimated to be $0,4$, $0,6$ and $0,8$, respectively. The efforts to

solve the problem will be intensfied and two more scientists will deal with the problem as members of some of the three teams. Table 5 shows for each (extended) team the estimated probability of its failure depending on the additonal number of scientists it gets assigned. Determine the best assignment of the two additional scientists to teams such that the estimated probability that all three teams fail is minimized. Solve this problem by means of dynamic programming. (Assume that the failures of the teams happen independently.)

Number of additional scientists assigned	Estimated failure probability		
	Team		
	1	2	3
0	0.4	0.6	0.8
1	0.2	0.4	0.5
2	0.15	0.2	0.3

Table 2: Data for Problem 5

6. The workload in a local company depends heavily on strong saisonal fluctuations. It is not desirable to dismiss a part of the staff in the periods with lower workload, and it is also not desirable to pay the wages of the peak-period all over the year, if this is not indispensable. Moreover, the management is principally against regular overtime hours. Since the production has to be done on demand it is not possible to build some inventory in the more quiet periods. In these circumstances it is difficult to determine an optimal employment policy.

   The estimated number of employees needed in the four seasons of the coming year is given as follows

Saison	Spring	Summer	Automn	Winter	Spring
Employees needed	255	270	240	200	255

   and the number of employees should in now way decrease under the above given levels. The extra costs for employing more people than (presumably) needed are estimated to be 2000 Euro per employee and season. Further it is assumed that the recruitement and dismission costs are given by the squared difference between the numbers of employees in the two consecutive seasons multiplied by 130. Notice that it is possible to employ part time staff; in this case the corresponding costs would be proportional to the emplyoment hours.

   The management wants to determine the levels of emplyement for every season of the following year such that the extra costs are minimized. Solve this problem by means of dynamic programming.

7. Formulate the following optimization problem as a deterministic dynamic programming problem with a finite time horizon (DDPPF) and solve it by applying the value iteration algorithm (cf. lecture).

$$\begin{array}{rl} \max & 3x_1 + 7x_2 + 6f(x_3) \\ \text{u.d.NB.} & \\ & x_1 + 3x_2 + 2x_3 \leq 6 \\ & x_1 + x_2 \leq 5 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

   where $f\colon [0, +\infty) \to \mathbb{R}$ is given as follows:

$$f(x) = \begin{cases} 0 & x = 0 \\ -1 + x_3 & x > 0 \end{cases}$$

8. Formulate the following optimization problem as a deterministic dynamic programming problem with a finite time horizon (DDPPF) and solve it by applying the value iteration algorithm (cf. lecture).

$$
\begin{array}{ll}
\max & z = x_1 x_2^2 x_3^3 \\
\text{udNB} & \\
& x_1 + 2x_2 + 3x_3 \leq 10 \\
& x_1 \geq 1,\ x_2 \geq 1,\ x_3 \geq 1 \\
& x_1,\ x_2,\ x_3 \text{ ganzzahlig}
\end{array}
$$