Advanced and algorithmic graph theory
Summer term 2016

6th work sheet (perfect graphs)

40. A graph is called *comparability graph* iff there exists a partial ordering of its vertex set such that two vertices are adjacent if and only if they are comparable. Show that every comparability graph is perfect.

41. A graph G is called an *interval graph* iff there exists a set $\{I_v : v \in V(G)\}$ of intervals such that $I_v \cap V_u \neq \emptyset$ if and only if $\{u, v\} \in E(G)$.

    (a) Show that every interval graph is chordal.

    (b) Show that the complement of every interval graph is a comparability graph.[1]

42. The Berge mistery story.

    Six professors had been to the library on the day that the precisous tractate was stolen. Each had entered once, stayed for some time and then left. If two professors were in the library at the same time, then at least one of them saw the other. Detectives questioned the professors and gathered the following testimony: Abe said that he saw Burt and Eddie in the library; Burt said that he saw Abe and Ida; Charlotte claimed to have seen Desmond and Ida; Desmond said that he saw Abe and Ida; Eddie testified to seeing Burt and Charlotte; Ida said that she saw Charlotte and Eddie. One of the professors lied!! Who was it?

43. Let σ be a perfect elimination scheme for a chordal graph G. Show that the greedy algorithm yields a $\chi(G)$-coloring of G if applied to the vertices ordered as $\sigma^{-1}(n)$, $\sigma^{-1}(n-2)$,. . ., $\sigma^{-1}(1)$.

44. Let G be a chordal graph.

    (a) Show that G has at most n maximum cliques (with respect to inclusion) and that equality holds iff $E(G) = \emptyset$.

    (b) Describe an efficient algorithm which identifies a clique of maximum cardinality in G and computes the clique number $\omega(G)$.

45. Consider a tree T and a collection of subtrees of T. The so-called *subtree graph* related to this collection of subtrees is a (so-called *intersection*) *graph* that has one vertex per subtree and an edge connecting any two subtrees that overlap in one or more nodes of the tree. Show that such a subtree graph is chordal. Is the converse also correct, i.e. given an arbitrary chordal graph G, is there a tree T and collection of subtrees of T such that the correspondig subtree graph is isomorphic to G?

46. A graph $G = (V, E)$ is called a *split graph* iff there is a partition $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ such that the induced subgraphs $G[V_1]$ and $G[V_2]$ are a clique and a stable (i.e. a graph with no edges), respectively.

    (a) Show that the class of split graph is closed under complementation, i.e. the complement of a split graph is a split graph.

    (b) Show that G is a split graph iff G and its complement $\bar{G}$ are both chordal.

[1] Conversely, a chordal graph is an interval graph if its complement is a comparability graph; this is a theorem of Gilmore and Hoffman (1964).

The fact that interval graphs have a reacher structure than general chordal graph or comparability graph has consequences for the theoretical complexity of certain prominent graph theoretical problems, e.g. the *Hamiltonian Cycle Problem*, i.e. the decision problem whether a graph has a Hamiltonian cycle or not, is NP-complete in general chordal graph [1] and also in bipartite graphs [3], but is however polynomially solvable in interval graphs [2].

References

[1] C.J. Colbourn and L.K. Stewart, Dominating cycles in series-parallel graphs, *Ars Combinatoria* **19A**, 185, 107–112.

[2] J.M. Keil, Finding Hamiltonian circuits in interval graphs, *Information Processing Letters* **20**, 1985, 201–206.

[3] M.S. Krishnamoorthy, An NP-hard problem in bipartite graphs, *SIGACT News* **7**, 1975, pp. 26.