Advanced and algorithmic graph theory
Summer term 2016

5th work sheet (colouring)

31. This example shows that the clique number $\omega(G)$ can be an arbitrarily bad lower bound on the chromatic number $\chi(G)$ of a graph G.

    Consider the sequence of graphs M_k, $k \in \mathbb{N}$, $k \geq 3$, constructed recursively as follows (cf. Mycielski 1955 [2]). Start with $M_3 := C_5$, the cycle on 5 vertices. The graph M_{k+1} is obtained from M_k by adding to M_k $(n+1)$ new vertices $u_1, u_2, \ldots, u_n, w$, where $n := |V(M_k)|$, such that w is connected to each u_i, $1 \leq i \leq n$, and u_i is connected to all vertices in $\Gamma(v_i)$, i.e. to all neighbours of v_i, $1 \leq i \leq n$. Show the following properties of M_k, $k \geq 3$:

    (i) M_k is triangle-free, i.e. it contains no cycle of length 3, $\forall k \geq 3$,

    (ii) $\chi(M_k) = k$,

    (iii) $|V(M_k)| = 3 \cdot 2^{k-2} - 1$.

    The graph M_4 is also called the Grötzsch-Graph [1].

32. This example shows that the quotient $\frac{|V(G)|}{\alpha(G)}$ can be an arbitrarily bad lower bound on the chromatic number $\chi(G)$ of a graph G with stability number $\alpha(G)$.

    Let G_k be a graph with $2k+1$ vertices such that $V(G_k) = V(K_k) \dot{\bigcup} V(S_k) \dot{\bigcup} \{w\}$, where K_k is an induced subgraph of G_k which is complete and has k vertices, S_k is an induced subgraph of G_k which has no edges and k vertices, and w is a vertex in G_k connected to all vertices of S_k and to no vertex of K_k. Moreover each vertex of S_k is connected to all vertices of K_k. Show that $\chi(G_k) = k+1$ and $\alpha(G_k) = k$ and deduce thereout the arbitrarily bad quality of the bound $\frac{|V(G)|}{\alpha(G)}$ for $\chi(G)$.

33. Calculate the chromatic number of a graph in terms of the chromatic number of its blocks.

34. (a) Show that every graph G has a vertex ordering for which the greedy algorithm only uses $\chi(G)$ colours.

    (b) For every $n \in \mathbb{N}$, $n \geq 2$, find a bipartite graph on $2n$ vertices ordered in such a way that the greedy algorithm uses n rather than 2 colours.

35. Find a graph G for which Brooks theorem yields a significantly weaker bound on $\chi(G)$ than the colouring number $col(G) := \max\{\delta(H) \colon H \subseteq G\} + 1$ (cf. lecture).

36. (a) Show that the complete graph K_n on n vertices is a class 1 graph iff n is an even number.

    (b) A 1-*factor* in a graph G is a perfect matching in G. A graph G is called 1-*factorisable* if its edge set can be partitioned into 1-factors. Show that regular graphs are class 1 graphs if and only if they are 1-factorisable.

37. A vertex list colouring of a graph is defined analogoulsy to the edge list colouring (cf. lecture). Given a graph G and lists of colours $L(v)$ for all $v \in V(G)$, a *vertex list colouring* is a mapping

    $$c \colon V(G) \to \bigcup_{v \in V(G)} L(v)\,, v \mapsto c(v)\,,$$

    such that $c(v) \in L(v)$, $\forall v \in V(G)$, and $c(u) \neq c(v)$ whenever $u, v \in V(G)$ and $\{u, v\} \in E(G)$ holds. G is called *vertex k-choosable* iff for any collection of lists $L(v)$, $v \in V(G)$, with $|L(v)| \geq k$, $\forall v \in V(G)$, there exists a vertex list colouring. The smallest natural number k for which a graph G is vertex k-choosable is called the *list chromatic number of* G (or *the choice number of* G) and is denoted by $\chi_l(G)$. Show that every plane graph is vertex 6-choosable[2].

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[1] M_4 is the unique smallest triangle-free 4-chromatic graph, where "smallest" refers to the number of vertices [1].

[2] There is a stronger result of Thomassen [3] stating that every planar graph is vertex 5-choosable.

38. For every natural number $k \in \mathbb{N}$ find a graph G with $\chi(G) = 2$ and $\chi_l(G) \geq k$.

39. Applications of coloring problems

    (a) Consider the following school timetabling problem. The dean is in charge of the timetable and has already assigned the courses to the teachers. Next he wants to assign a time slot from a given set of time slots to every course (e.g. 10 teaching units per day, 5 days per week). Model this problem as a graph coloring problem by assuming that a teacher can teach at most one course at a time and a class can take at most one course at a time. Assume moreover that teachers might not be available at all slots. How would you modify your model to determine a valid assignment of courses to the time slots in this case?

    (b) Model the sudoku game as a graph coloring problem. Consider that a typical sudoku has also some already filled cells.

References

[1] V. Chvátal, The minimality of the Mycielski graph, in *Graphs and Combinatorics*, *Lecture Notes in Mathematics* **406**, 243–246, Springer, Berlin, 1973.

[2] J. Mycielski, Sur le coloriage des graphes, *Colloq. Math. 3*, 1955, 161–162.

[3] C. Thomassen, Every planar graph is 5-choosable, *Journal of Combinatorial Theory (B)* **62(1)**, 1994, 180–181.