Advanced and algorithmic graph theory
Summer term 2016

3d work sheet

20. Consider the following theorem of Chvátal and Erdös (cf. lecture)

    Let G be a graph with at least 3 vertices ($|V(G)| \geq 3$), connectivity number $\kappa(G)$ and stability number $\alpha(G)$. If $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian.

    Show that this theorem is best possible, in the sense that there exist non-hamiltonian graphs G with $\alpha(G) = \kappa(G)+1$. Illustrate this fact for the Petersen graph and the complete bipartite graph $K_{r,r+1}$, $r \in \mathbb{N}$.

21. Consider the following theorem of Ore, Bermond and Linial (cf. lecture)

    Let G be a 2-connected graph in which $d(x) + d(y) \geq d$ holds for any two non-adjacent vertices $x, y \in V(G)$ and some arbitrary but fixed natural number d. Then there exists a cycle of length at least $\min\{n, d\}$ in G.

    Show that this theorem directly implies the following two results

    (1) If in a graph G with $|V(G)| \geq 3$, the inequality $d(x) + d(y) \geq n$ holds for any two vertices $x, y \in V(G)$ such that $\{x, y\} \notin E(G)$, then G is hamiltonian. (Due to Ore, cf. lecture.)

    (2) A graph G with $n := |V(G)| \geq 3$ and minimum degree $\delta(G) \geq n/2$ is hamiltonian. (Due to Dirac, cf. lecture.)

22. Let G be a connected graph with $d := \min\{d(x) + d(y)\colon x, y \in V(G), \{x, y\} \notin E(G)\}$. If $d \geq 2\delta(G)$ holds, then G contains a path with at least $\min\{d + 1, n\}$ vertices.

23. Construct

    (a) a non-hamiltonian connected 4-regular graph with 11 vertices, and

    (b) a non-hamiltonian 2-connected 4-regular graph.

24. Show that the d-dimensional cube Q_d is hamiltonian (cf. Exercise No. 2 for the definition of Q_d).

25. A graph G is called *hamiltonian connected* iff for any two different vertices $u, v \in V(G)$, $u \neq v$, there exists a hamiltonian u-v-path in G. Prove the following statements:

    (a) If $d := \min\{d(x) + d(y)\colon x, y \in V(G), \{x, y\} \notin E(G)\} \geq |V(G)| + 1$ holds, than also $\kappa(G) \geq \alpha(G) + 1$ holds.

    (b) If $\kappa(G) \geq \alpha(G) + 1$ holds for some graph G, then G is hamiltonian connected.

    (c) If the $(n + 1)$-st hamiltonian hull $\mathcal{H}_n(G)$ of G is isomorphic to K_n, then G is hamiltonian connected.

26. Show that a planar graph is 2-connected, iff the border of every region is a cycle.