Advanced and algorithmic graph theory
Summer term 2016

2nd work sheet

14. Prove the following theorem stated in the lecture.

    Let G be a connected graph. Assume that $DFS(s)$, i.e. a depth first search starting at some $s \in V(G)$, has been performed in G. Consider the classification of edges from $E(G)$ into *tree edges* (collected in the set T) and *backward edges* (collected in the set B), as well as their orientation according to $DFSNum$ (cf. lecture). This orientation allows the specification of a starting vertex and an end vertex for every edge (cf. lecture). For all $v \neq s$ and for all $\{v, w\} \in E(G)$ starting at v the following holds: $\{v, w\}$ and the tree edge ending at v belong to the same block if and only if one of the following conditions holds: (a) $\{v, w\}$ is a backward edge or (b) $\{v, w\}$ is a tree edge which is not a leading edge.

15. Prove the following theorem stated in the lecture.

    Let G be a connected graph. Assume that $DFS(s)$, i.e. a depth first search starting at some $s \in V(G)$, has been performed in G. The following statements hold:

    (a) The root s is a cut-vertex if and only if there exists more than one leading edge incident to s.

    (b) A vertex $v \in V(G) \setminus \{s\}$ is a cut-vertex if and only if there exists at least one leading edge starting at v.

16. Let G be a $2k$-edge connected graph for some $k \in \mathbb{N}$. Show that G contains at least k edge-disjoint spanning trees. Is this result best possible, i.e. is there any $2k$-edge connected graph, which does not contain $k+1$ edge-disjoint spanning trees, for some $k \in \mathbb{N}$? Given an arbitrary $k \in \mathbb{N}$, can you find a $2k$-edge connected graph, which does not contain $k+1$ edge-disjoint spanning trees?

17. A graph G is called *cubic*, if all vertices of G have degree 3, i.e. $d_G(v) = 3$, for all $v \in V(G)$. Show that for a cubic graph G the equality $\lambda(G) = \kappa(G)$ holds, i.e. the vertex connectivity equals the edge connectivity.

18. (a) Show that for a graph G with $diam(G) = 2$ the equality $\lambda(G) = \delta(G)$ holds.

    (b) Let G be a graph with $|V(G)| \geq 2$ such that $d(u) + d(v) \geq n - 1$ holds, for all $u, v \in V(G)$ with $\{u, v\} \notin E(G)$. Show that $\lambda(G) = \delta(G)$.

19. (a) Show that for the d-dimensional cube Q_d, $d \in \mathbb{N}$, $d \geq 2$, the equality $\kappa(Q_d) = \delta(Q_d) = d$ holds. (See Exercise No. 2 for the definition of Q_d.)

    (b) A *Halin graph* H is defined as a graph obtained from a tree T without vertices of degree 2 by adding to it a cycle which joins all the leaves of T. Show that $\kappa(H) = \delta(H) = 3$ holds for any Halin graph H.