Advanced and algorithmic graph theory
Summer term 2016

1st work sheet

1. Show that $rad(G) \leq diam(G) \leq 2rad(G)$ hold for every graph G, where $rad(G)$ denotes the radius of graph G and $diam(G)$ denotes its diameter as defined in the lecture.

2. Let $d \in \mathbb{N}$ and $V = \{0,1\}^d$, thus V is the set of all 0-1-sequences of length d. The graph with vertex set V in which two such sequences form an edge if and only if they differ in exactly one position, is called the d-dimensional cube. Determine the average degree, the number of edges, the diameter, the girth and the circumference of this graph.

   (Hint for the circumference: induction on d.)

3. Prove that a graph G with minimum degree $\delta := \delta(G)$ and girth $g := g(G)$ has at least $n_0(\delta, g)$ vertices, where
$$n_0(\delta, g) := \begin{cases} 1 + \delta \sum_{i=0}^{r-1} (\delta - 1)^i & \text{if } g =: 2r+1 \text{ is odd} \\ 2 \sum_{i=0}^{r-1} (\delta - 1)^i & \text{if } g =: 2r \text{ is even} \end{cases}$$

4. Determine the connectivity $\kappa(G)$ and the edge connectivity $\lambda(G)$ for (a) $G = P_m$ being a path of length m, (b) $G = C_n$ being a cycle of length n, (c) $G = K_n$ being a complete graph with n vertices, (d) $G = K_{m,n}$ being a complete bipartite graph with m and n vertices in its partition sets, respectively, and (e) G being the d dimensional cube.

5. Prove the following theorem of Dirac (1960): Any k vertices of a k-connected graph, $k \geq 2$, lie on a common cycle.

6. Show that a graph G is 2-edge connected if and only if it possesses a *weak ear decomposition*, i.e. G can be obtained as $G := G_0 \cup G_1 \cup G_2 \cup \ldots G_k$, where G_0 is a cycle and every graph G_i is either a path which has only the two end-vertices in common with $V(G_0 \cup G_1 \cup \ldots G_{i-1})$, or G_i is a cycle which has just one vertex in common with $G_0 \cup G_1 \cup \ldots G_{i-1}$, for $1 \leq i \leq k$.

7. (s-t-labelling)
   Let $G = (V, E)$ be a graph and $\{s,t\} \in E$. Show that the following holds: G is 2-connected if and only if there exists a bijective mapping $\sigma\colon V \to \{1, 2, \ldots, n := |G|\}$ (called *s-t-labelling*), such that $\sigma(s) = 1$, $\sigma(t) = n$, and for every $v \in V \setminus \{s,t\}$ there exist two neighbors $x, y \in N(v)$ with $\sigma(x) < \sigma(v) < \sigma(y)$.

8. A *block* of a graph G is a maximal connected subgraph without a cut-vertex. Show that, if G is connected, then the central vertices of G (cf. the lecture for the definition) lie on a block of G.

9. Let $G = (V, E)$ be a graph and $\sim$ be a binary relation defined on E such that $e_1 \sim e_2$ if and only if $e_1 = e_2$ or e_1 and e_2 lie on a common cycle in G. Show that $\sim$ is an equivalence relation and that the equivalence classes of $\sim$ are exactly the edge sets of the blocks of G. An edge e forms as a singleton an equivalence class $\{e\}$ of $\sim$ iff e is a bridge in G (cf. the lecture for the definition of a bridge).

10. (Normal trees) A tree T with a fixed vertex r in T is called *a tree rooted at* r. Consider the relation $\preceq$ in $V(T)$ associated with T and r defined as follows: $x \preceq y$ iff $x = y$ or x lies in the unique path r-y-path in T. (We can consider this also as a "height" relation and say that x lies below y in T iff $x \preceq y$ and $x \neq y$. We say that the vertices of T at distance k from r have height k and form the *kth level of* T.) Further denote the *down-closure* $\lceil y \rceil$ of y and the *up-closure* $\lfloor x \rfloor$ of X as follows:
$$\lceil y \rceil := \{x\colon x \preceq y\} \text{ and } \lfloor x \rfloor := \{y\colon x \preceq y\}\,, \text{ respectively.}$$

    Show that

(a) $\preceq$ is a partial order in $V(T)$.

(b) The root r is the *least* (or *minimum*) *element* in $\prec$.

(c) The leaves of T are *maximal elements* in $\prec$.

(d) The end-vertices of eny edge in $E(T)$ are comparable in $\preceq$.

(e) The down closure of each veretx in $V(T)$ is a *chain*, i.e. a set of pairwise comparable elements.

11. Let $G = (V, E)$ be a graph and let T be a subgraph of G which is a rooted tree with root $r \in V(T)$. T is called *normal* in G iff the end-vertices of every $V(T)$-path in G are comparable with respect to the relation $\preceq$ associated with T and r (c.f. Exercise no. 10). Show that the following holds for any normal tree T in G

    (a) Any two vertices $x, y \in V(T)$ are separated in G by the set $\lceil x \rceil \cap \lceil y \rceil$.

    (b) If $S \subseteq V(T) = V(G)$ and S is down-closed (i.e. S contains the down-closure of any element $s \in S$), then the components of $G - S$ are spanned by the sets $\lfloor x \rfloor$ with x minimal in $V(T) - S$.

12. Let G be a connected graph and let $r \in V(G)$. Show that there exists a normal spanning tree T rooted at r in G.

13. Consider some ear decomposition of a 2-connected graph $G = (V, E)$ (cf. the lecture for its definition) and show that the number of ears equals $|E| - |V|$.