A second look at binary digits

Christiaan van de Woestijne

(Supported by FWF Project S9606)

Institut für Mathematik B

Technische Universität Graz

Lehrstuhl für Mathematik und Statistik

Montanuniversität Leoben

ÖMG-DMV-Kongreß 2009, Graz

21–25 September 2009

Binary digits

Everybody knows that $(10)_2 = 2$ and $(11011)_2 = 27$.

Also, $-27 = (-11011)_2$. Or is it?

Some computers know that $-27 = (1111111111100101)$ (signed word), and that $32767 + 1 = -32768$.

Some people know that $-1 = (11111\ldots)_2 \in \mathbb{Z}_2$ (start with LSD here), and

$$-27 = (101001111\ldots)_2.$$

Can we do better?

The expansion algorithm

Define the dynamic mapping $T : \mathbb{Z} \to \mathbb{Z} : a \mapsto \begin{cases} \frac{a}{2} & \text{if } a \text{ even;} \\ \frac{a-1}{2} & \text{if } a \text{ odd.} \end{cases}$

Now to expand a, write 0 if a even and 1 otherwise, and continue with $T(a)$. Done when $T^n(a) = 0$.

Example: $27 \xrightarrow{1} 13 \xrightarrow{1} 6 \xrightarrow{0} 3 \xrightarrow{1} 1 \xrightarrow{1} 0$.

However, $-1 \xrightarrow{1} -1 \ldots$

Try other digits: $\mathcal{D} = \{d_0, d_1\}$, with $d_i \equiv i \pmod 2$.

Criterion for the existence of a 1-cycle: $\frac{a-d}{2} = a \Leftrightarrow a = -d$.
So this is hopeless!

Negabinary expansions

Try other basis -2, with digits $\{0, 1\}$:

$-27 \xrightarrow{1} 14 \xrightarrow{0} -7 \xrightarrow{1} 4 \xrightarrow{0} -2 \xrightarrow{0} 1 \xrightarrow{1} 0$, so $-27 = (100101)_{-2}$.

Theorem (Grünwald 1885) All integers have a finite expansion on the integer basis $b \leq -2$ and digits $\{0, 1, \ldots, |b| - 1\}$.

Proof: there are no cycles except $0 \xrightarrow{0} 0$!

Excursion: the balanced ternary expansion uses basis $+3$ and digits $\{-1, 0, 1\}$, and expands all integers finitely. If only computers had three-way switches!

Theorem Let $a \in \mathbb{Z}_3$. Then $a \in \mathbb{Z}$ if and only if its balanced ternary expansion is finite.

A curious question

Definition A digit set $\mathcal{D}$ is valid for basis ± 2 if all integers have a finite representation

$$\sum_{i=0}^{\ell} d_i(\pm 2)^i \quad (d_i \in \mathcal{D}).$$

We know that no digit sets are valid for basis $+2$; for basis -2, we know the valid digit set $\{0, 1\}$, and thus also $\{0, -1\}$ by an automorphism of the additive group.

Question Are there any others?

Answer Yes, infinitely many!

Expansions of zero

Is it possible to have a digit set without zero? Yes!

The definition of the mapping T and of the stopping criterion is the same (if you formulate it like I do!).

Example: basis -2, digits $\{1, 4\}$. Expand -27:

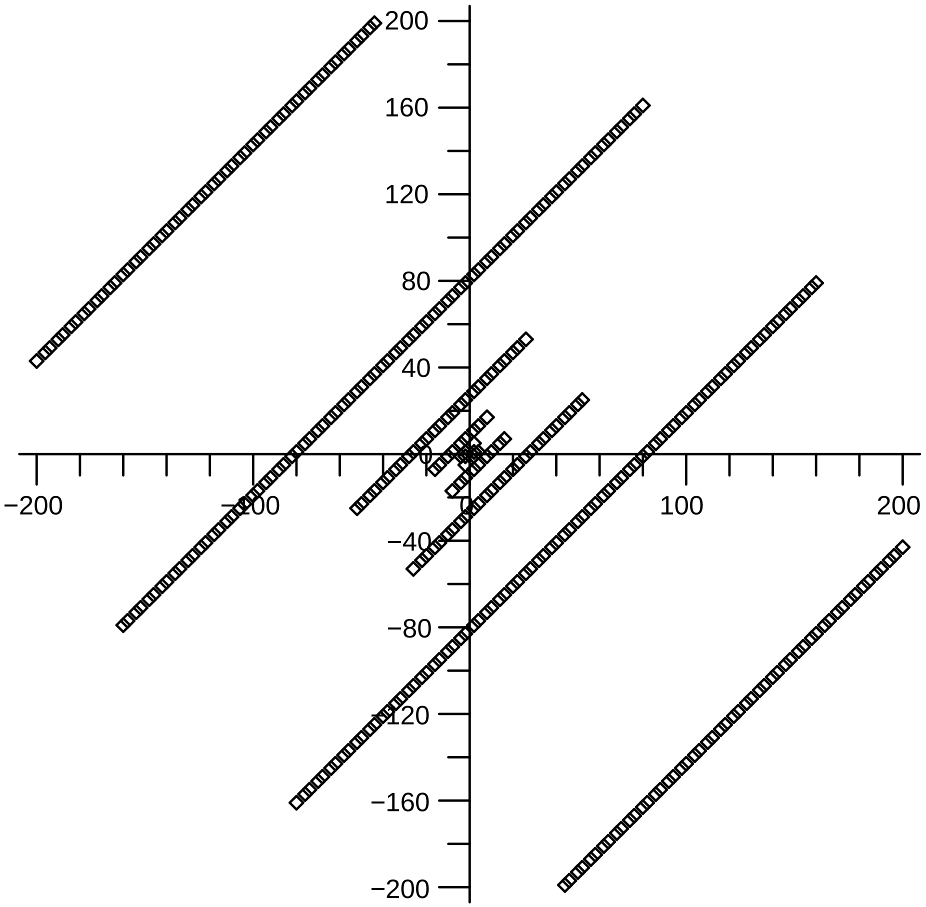
$-27 \xrightarrow{1} 14 \xrightarrow{4} -5 \xrightarrow{1} 3 \xrightarrow{1} -1 \xrightarrow{1} 1 \xrightarrow{1} 0$, so $-27 = (111141)_{-2}$.

Interesting: $0 \xrightarrow{4} 2 \xrightarrow{4} 1 \xrightarrow{1} 0$, a 3-cycle!

So, $0 = ()_{-2} = (144)_{-2} = (144144)_{-2} = \dots$

Theorem Any valid digit set gives rise to a nontrivial expansion of zero.

Experiments



The figure plots all pairs of integers (x, y), with $|x|, |y| \leq 200$, that are valid digit sets for basis -2.

Results

Theorem The digit set $\{d, D\}$ with $d < D$ is valid for basis -2 if and only if

(i) one of d, D is even and one is odd (trivial)

(ii) either $dD = 0$ or $3 \nmid dD$ (avoid 1-cycles except 0)

(iii) we have $2d \leq D$ and $2D \geq d$ (0 is expansible)

(iv) $D - d = 3^i$ for some $i \geq 0$ (the real stuff!)

For example, the only valid digit sets with 0 are $\{0, \pm 1\}$. On the other hand, the sets $\{1, 3^i + 1\}$ are valid for all $i \geq 0$.

Higher-dimensional analogues

There is no reason to limit the theory of number systems to $\mathbb{Z}$. Consider this setup:

- $\mathcal{O}$ is a $\mathbb{Z}$-order.
- $\alpha \in \mathcal{O}$ is nonzero.
- $\mathcal{D}$ represents $\mathcal{O}$ modulo α (we have $|\mathcal{D}| = |\operatorname{Norm}(\alpha)| < \infty$).

Then we can define $T : \mathcal{O} \to \mathcal{O} : a \mapsto \frac{a-d_a}{\alpha}$, where $d_a \in \mathcal{D}$ has $a \equiv d_a \pmod{\alpha}$.

Easy necessary conditions to have finite expansibility of all $a \in \mathcal{O}$:

- α and $\alpha - 1$ must be non-units of $\mathcal{O}$
- α must be expanding, i.e., for all $\sigma : \mathcal{O} \hookrightarrow \mathbb{C}$ we have $|\sigma(\alpha)| > 1$.

The periodic set

With this setup, we call $(\mathcal{O}, \alpha, \mathcal{D})$ a pre-number system.

Because α is expanding, the mapping T is almost a contraction on $\mathcal{O}$, and the unique finite subset $\mathcal{P} \subset \mathcal{O}$ that is invariant under T is called the periodic set of the pre-number system.

Theorem The periodic set of $(\mathbb{Z}, -2, \{d, D\})$ is the arithmetic progression $\left\{\left\lceil \frac{2d-D}{3} \right\rceil, \ldots, \left\lfloor \frac{2D-d}{3} \right\rfloor\right\}$.

In higher dimensions, the geometric structure of the periodic set is quite complicated.

The tile

There is a continuous variant of the (discrete) periodic set, called the tile of the pre-number system, because it usually tiles $\mathcal{O} \otimes \mathbb{R}$.

For $(\mathbb{Z}, -2, \{d, D\})$, it is the interval $\left[\frac{2d-D}{3}, \frac{2D-d}{3}\right]$.

In higher dimensions, these tiles usually have fractal boundary.

To prove a higher-dimensional analogue of the main Theorem, we must characterise the lattice points in the tile, and describe the action of T on them.

Work in progress

More-or-less-theorem Let α be an expanding algebraic integer of norm ± 2. Then up to finitely many exceptions, a digit set $\mathcal{D} = \{d_0, d_0 + \delta\}$ makes $(\mathbb{Z}[\alpha], \alpha, \mathcal{D})$ into a number system if and only if:

(i) $\delta \equiv 1 \pmod{\alpha}$ and $(d_0, \alpha - 1) = (1)$

(ii) there is a nontrivial zero expansion

(iii) δ is a product of prime divisors of $\alpha - 1$ that are regular, totally split and lie over different primes of $\mathbb{Z}$

Note that for a given degree d, there are only finitely many expanding α of degree d and norm ± 2. A famous example is $\tau = \frac{-1+\sqrt{-7}}{2}$ satisfying $x^2 + x + 2$. The smallest nonmaximal order among them is generated by $x^4 + x^2 + 4$ (Potiopa 1997). The smallest example with a nontrivial ideal class group is $x^8 - x^6 - x^2 + 2$ (CvdW 2009).