Generalised binary number systems

Christiaan van de Woestijne
Institut für Mathematik B
Technische Universität Graz, Austria

Supported by the FWF, project S9606

Montanuniversität Leoben
27 February 2009

Abstract

Let A be a square integer matrix of determinant ± 2, and assume A is expanding, that is, all its eigenvalues are greater than 1 in absolute value. Let $\{d, D\}$ be integer vectors such that d is in the image of A and D is not. If every integer vector v has a representation of the form

$$v = d_0 + Ad_1 + A2d_2 + \ldots + A^k d_k$$

with the d_i being either d or D, we call the triple $(A, \mathbb{Z}^n, \{d, D\})$ a number system.

Our goal, which will not be achieved in this talk, is to classify all such number systems with two digits, which generalise the well-known binary number system. We will show the technical obstacles for such a classification and also give some partial results, such as a complete classification in the 1-dimensional case.

Definitions

We define a pre-number system as a triple $(V, \phi, \mathcal{D})$, where

- V is a finite free $\mathbb{Z}$-module;
- ϕ is an expanding endomorphism of V;
- $\mathcal{D}$ is a system of representatives of V modulo $\phi(V)$.

A pre-number system $(V, \phi, \mathcal{D})$ is a number system if there exist finite expansions

$$a = \sum_{i=0}^{\ell} \phi^i(d_i) \qquad (d_i \in \mathcal{D})$$

for all $a \in V$.

We are ultimately interested in the classification of all number systems.

Examples

- $(\mathbb{Z}, b, \{0, \ldots, |b|-1\})$ is a pre-number system whenever $|b| \geq 2$, and a number system if and only if $b \leq -2$.

- $(\mathbb{Z}[i], b, \{0, \ldots, |b|^2-1\})$ is a pre-number system whenever $|b| > 1$, and a number system if and only if $b = -a \pm \mathrm{i}$, for some $a \in \mathbb{N}$.

- $(\mathbb{Z}, -2, \{d, D\})$ is a number system if and only if ... (see later)

- $(\mathbb{Z}[X]/((X-5)(X-7)), X, \{1, -1, 3, -3, 5, X, X-2, -X+2, X-4, -X+4, X-6, -X+6, X-8, -X+8, -X+10, 2X-7, 2X-9, -2X+9, 2X-11, -2X+11, 2X-13, -2X+13, -2X+15, 3X-14, 3X-16, -3X+16, -3X+18, 3X-18, -3X+20, 4X-21, 4X-23, -4X+23, -4X+25, 5X-28, -5X+30\})$ is a number system

Example: the odd digits

Assume $V = \mathbb{Z}$ and ϕ is multiplication by some integer b. Let b be odd, $|b| \geq 3$, and let

$$\mathcal{D}_{\text{odd}} := \{-|b| + 2, -|b| + 4, \ldots, -1, 1, \ldots, |b| - 2, b\}.$$

This is a valid digit set for all odd b.

For $b = 3$: it's $\{-1, 1, 3\}$. We get $0 = 3 \cdot 1 + (-1) \cdot 3$.

a	$(a)_{3,\text{odd}}$	a	$(a)_{3,\text{odd}}$	a	$(a)_{3,\text{odd}}$	a	$(a)_{3,\text{odd}}$
0	$\overline{1}3$	5	$1\overline{1}\overline{1}$	-1	$\overline{1}$	-6	$\overline{1}133$
1	1	6	13	-2	$\overline{1}1$	-7	$\overline{1}1\overline{1}$
2	$1\overline{1}$	7	$1\overline{1}1$	-3	$\overline{1}13$	-8	$\overline{1}131$
3	3	8	$3\overline{1}$	-4	$\overline{1}\overline{1}$	-9	$\overline{1}\overline{1}3$
4	11	9	$1\overline{1}3$	-5	$\overline{1}11$	-10	$\overline{1}13\overline{1}$

The dynamic mapping

Define functions

$$d : V \to \mathcal{D} : d(a) \text{ is the unique } d \in \mathcal{D} \text{ with } a - d \in \phi(V);$$
$$T : V \to V : T(a) = \phi^{-1}(a - d(a)).$$

We call T the dynamic mapping of $(V, \phi, \mathcal{D})$.

Theorem $(V, \phi, \mathcal{D})$ is a number system if and only for all $v \in V$ there exists $n \geq 0$ with $T^n(v) = 0$.

Recall that a pre-number system has a finite attractor $\mathcal{A} \subseteq V$ with the properties

- for all $a \in V$ we have $T^n(a) \in \mathcal{A}$ if n is large enough.
- T is bijective on $\mathcal{A}$.

Theorem $(V, \phi, \mathcal{D})$ is a number system if and only if the attractor contains 0, and consists exactly of one cycle under T.

Tiles and translation

The tile of the pre-number system $(V, \phi, \mathcal{D})$ is

$$\mathcal{T} = \left\{ \sum_{i=1}^{\infty} \phi^{-i}(d_i) : d_i \in \mathcal{D} \right\}.$$

By results of Lagarias and Wang (building on earlier authors), $\mathcal{T}$ is a compact set of positive measure that is the closure of its interior (show many examples). Let Λ be the $\mathbb{Z}[\phi]$-submodule of V generated by $\mathcal{D} - \mathcal{D}$, the differences of the digits; then we can tile $V \otimes \mathbb{R}$ with $\mathcal{T}$ by a sublattice M of Λ, and we have

$$\mu(\mathcal{T}) = [V : M] = [\Lambda : M] \cdot [V : \Lambda].$$

If the characteristic polynomial of ϕ is irreducible, then we may take $\Lambda = M$.

One can prove that the attractor $\mathcal{A}$ is equal to $-\mathcal{T} \cap V$.

Binary number systems

Suppose $|\det(\phi)| = 2$; then there are exactly 2 digits, and we speak of a binary (pre-)number system. There are many special properties:

- The tile is connected
- The characteristic polynomial χ_ϕ is irreducible
- The tiling lattice is generated by one element

We may assume V is an ideal in $R = \mathbb{Z}[\alpha]$, where α is a zero of $f = \chi_\phi$.

Write $\mathcal{D} = \{d, D\}$ with d divisible by α in V and D not, and let

$$\delta = d - D.$$

Then the tiling lattice is the ideal generated by δ, and $\mu(\mathcal{T}) = |\,\mathrm{Norm}(\delta)|$.

The goal

We want to classify all binary number systems, that is, given an algebraic integer α of norm ± 2 and an ideal $V \subseteq \mathbb{Z}[\alpha]$, find all pairs $\{d, D\}$ such that $(V, \alpha, \{d, D\})$ is a number system.

To do this, we have two tasks:

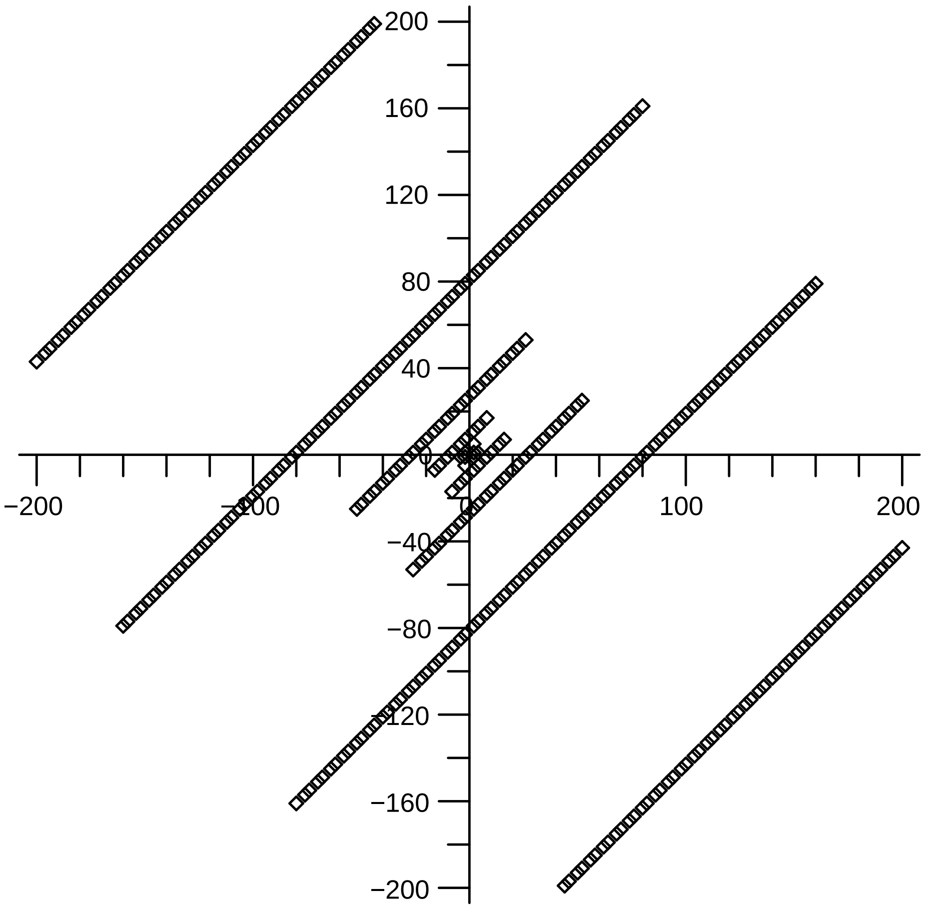
1. compute how many elements are in $\mathcal{A}$;
2. find their cycle structure under the dynamic map T.

Note that if all points of $\mathcal{A}$ are interior points of $-\mathcal{T}$, then $|\mathcal{A}| = |\operatorname{Norm}(\delta)|$, since only boundary points can be in more than one tile translate.

Note also that when $\alpha - 1$ is a unit, then $d/(\alpha - 1)$ and $D/(\alpha - 1)$ both start 1-cycles in $\mathcal{A}$, so α is not the base of any number system.

Example: the case $V = \mathbb{Z}$

Let $V = \mathbb{Z}$; then $\alpha = \pm 2$. If $\alpha = 2$, then $\alpha - 1$ is a unit.



In the figure, we see all valid digit sets for $\alpha = -2$ with both digits less than 200 in absolute value.

What is the structure of this set?

The fundamental lemma

We are interested in the cycles in V under the dynamic map T. Now $a_0 \in V$ starts a cycle of length ℓ if and only if

$$a_0(1-\alpha^\ell) = \sum_{i=0}^{\ell-1} d_i \alpha^i.$$

Now because the only digits are d and $d-\delta$, this means

$$a_0(1-\alpha^\ell) = d\frac{\alpha^\ell - 1}{\alpha - 1} - \delta \sum_{i=0}^{\ell-1} \varepsilon_i \alpha^i,$$

so that

$$(d + (\alpha-1)a_0)\frac{\alpha^\ell - 1}{(\alpha-1)\delta} = \sum_{i=0}^{\ell-1} \varepsilon_i \alpha^i,$$

with $\varepsilon_i = 0, 1$ for all i.

This is our fundamental tool to study the cycle structure.

Algebraic number theory

Theorem Suppose $\delta = \prod \pi_i^{h_i}$, where the π_i are regular totally split primes of $\mathbb{Z}[\alpha]$ dividing $\alpha - 1$ lying above distinct primes of $\mathbb{Z}$, and such that π_i divides $\alpha + 1$ exactly once if π_i lies above 2. Then

$$(\alpha - 1)\delta \text{ divides } \alpha^\ell - 1$$

if and only if $\mathrm{Norm}(\delta)$ divides ℓ.

Conversely, if the order of α modulo $(\alpha - 1)\delta$ is $|\mathrm{Norm}(\delta)|$, and δ is made up of regular primes, then up to a factor of bounded norm, δ is as described above.

I do not know if it is necessary for α to have order $|\mathrm{Norm}(\delta)|$ for δ large enough, in order to have a number system, but my examples lead me to conjecture that it is.

Sketch of proof

Suppose δ has the right form. Let π^h exactly divide δ. As π divides $\alpha - 1$, the order of α modulo π^g is 1, where $g = v_\pi(\alpha - 1)$. If π is regular and unramified and lies above p, then

$$\pi^{g+i} \parallel \alpha^{p^i} - 1.$$

Thus if also π has residue degree 1, we have $\mathrm{Norm}(\pi) = p$ and

$$\alpha^{|\,\mathrm{Norm}(\pi^h)|} \equiv 1 \pmod{(\alpha - 1)\pi^h},$$

where the exponent is minimal with this property.

Combining divisors of δ, the order of α modulo $\prod \pi_i^{h_i}$ is the l.c.m. of those modulo the $\pi_i^{h_i}$.

If π lies over p with ramification index e and residue class degree f, then the order of α modulo $(\alpha - 1)\pi^h$ is roughly $p^{h/e}$, whereas the norm of π^h is p^{fh}. Thus, we want $e = f = 1$.

Points in the tile

Given δ, first compute the order ℓ of α modulo $(\alpha - 1)\delta$. Then, we know that the length of every cycle in $\mathcal{A}$ is divisible by ℓ. Thus, ℓ divides $|\mathcal{A}| = |\mathcal{T} \cap V|$.

Note that $\ell = 1$ if and only if δ is a unit.

If we embed $\mathbb{Z}[\alpha]$ into $\mathbb{R}^n$ using the canonical embedding, then $\mathcal{T}$ is equal to the tile corresponding to the digit set $\{0, 1\}$ multiplied by δ (a diagonal linear map).

Conjecture If $\ell = |\,\mathrm{Norm}(\delta)|$ and ℓ is large enough, then $|\mathcal{A}| = \ell$. Equivalently, then all lattice points of $\mathcal{T}$ are interior.

If the Conjecture is false, we can have huge numbers of lattice points on the tile boundary. Note that δ need not be expanding.

Examples: factorisation of $\alpha - 1$

If $\alpha = 2$, then $\alpha - 1 = 1$, a unit. If $\alpha = -2$, then $\alpha - 1 = -3$, so the only prime dividing $\alpha - 1$ is 3.

If $f = x^4 + x + 2$, then $\alpha - 1 = (\alpha + 1)^2$, where $\alpha + 1$ is a totally split prime lying over 2. This implies that for $f = x^4 - x + 2$, we have $\alpha + 1 \sim (\alpha - 1)^2$!

If $f = x^4 + x^3 + 2x^2 + x + 2$, then $\alpha - 1$ is a totally split prime lying over 7. However, if $\{d, D\} = \{\alpha, 1\}$, $\mathcal{A}$ consists of a cycle of length 14, with elements pairwise congruent modulo $\alpha - 1$. If $\{d, D\} = \{\alpha^2 - 2, 2\alpha - 3\}$, we have $\delta = (\alpha - 1)^2$ and, indeed, $\mathcal{A}$ has one cycle of length 49.

If $f = x^4 + x^2 + x + 2$, then for $\{d, D\} = \{0, 1\}$, we have an 11-cycle! For digits $\{\alpha, 1\}$, we have two 5-cycles, one containing 0 and the other $\alpha - 1$. For digits $\{\alpha^2 + 2\alpha + 2, 1\}$, with $\delta = (\alpha - 1)^2$, we find a unique cycle of length 25.

Examples: factorisation of $\alpha - 1$ (2)

Among all expanding $f \in \mathbb{Z}[x]$ with degree at most 8 and $|f(0)| = 2$, the only prime divisors of $\alpha - 1$ with residue degree more than 1 are non-regular.

However, many primes are ramified. For example, for $f = x^5 - x + 2$, $\alpha - 1$ lies over 2 with ramification index 4! We find, for example, that $\alpha^8 - 1$ is divisible by $(\alpha - 1)^9$, which has norm 2^9, whereas we would like to have only 3 factors, with norm 8.

An interesting case is $f = x^2 + x + 2$, with root $\tau = \frac{-1+\sqrt{-7}}{2}$, which is much used in cryptography. Here, $\tau - 1 = (\tau + 1)^2$, and $\tau + 1$ is a regular prime of norm 2. Thus, all conditions on τ are met.

Indeed, I have computed all valid digit sets for base τ of the form $\{a + b\tau, c + d\tau + 1\}$ with $a, b, c, d \in \{-4, \ldots, 4\}$, and it turns out that for all of them, δ is a power of $\tau + 1$. All attractors have the "right" number of elements, except when δ is a unit and $dD \neq 0$; in those cases, $|\mathcal{A}| = 3$.

Example: the case $V = \mathbb{Z}$ (2)

Let $\alpha = -2$, let $\delta \in \mathbb{Z}$ odd with $|\delta| > 1$; let $d, D \in \mathbb{Z}$ with $2 \mid d$ and $D = d - \delta$. We have:

1. $|\mathcal{A}| = |\delta|$ iff $3 \nmid dD$;
2. if $3 \nmid dD$, then $\mathcal{A}$ has one cycle if and only if $|\delta| = 3^i$ with $i \geq 1$; if $3 \mid dD$, then $\mathcal{A}$ has more than one cycle;
3. there is an easy criterion to see whether $0 \in \mathcal{A}$.

In fact, the only connected subsets of $\mathbb{R}$ are intervals, so $\mathcal{T}$ must be an interval.

If $|\delta| = 1$, then the only valid $\{d, D\}$ are $\{0, \pm 1\}$, $\{1, 2\}$ and $\{-1, -2\}$. For the latter, $\mathcal{T}$ has only boundary lattice points.

Main theorem

Let α be an expanding algebraic integer of norm ± 2, and suppose $\delta = \prod \pi_i^{h_i}$ where the π_i are regular totally split primes of $\mathbb{Z}[\alpha]$ dividing $\alpha - 1$ and lying above distinct primes of $\mathbb{Z}$, and such that π_i exactly divides $\alpha + 1$ if π_i lies above 2.

Let $d, D \in \mathbb{Z}[\alpha]$ have $d - D = \delta$, let $V = (d, D)$, and suppose that $d \in \alpha V$ (so that $D \notin \alpha V$, because α is prime).

Let $\mathcal{T}$ be the tile of $(V, \alpha, \{d, D\})$, and suppose $\mathcal{T} \cap V$ consists of interior points of $\mathcal{T}$, and that $0 \in \mathcal{T}$.

Then $(V, \alpha, \{d, D\})$ is a number system.

I conjecture that the converse holds: if Norm(δ) is large enough, and $(V, \alpha, \{d, D\})$ is a number system, then δ has the form given above and all points of $\mathcal{T} \cap V$ are interior.

Example: the case $V = \mathbb{Z}$ (3)

Theorem Let $d, D \in \mathbb{Z}$, with $d < D$. Then $(\mathbb{Z}, -2, \{d, D\})$ is a number system if and only if

1. one of $\{d, D\}$ is even and one is odd;
2. neither of d and D is divisible by 3, except when the even digit is 0;
3. we have $2d \leq D$ and $2D \geq d$;
4. $D - d = 3^i$ for some $i \geq 0$.

Example Thus, $\{1, 3^k + 1\}$ is valid for $b = -2$, for all $k \geq 0$.

The only valid digit sets for $b = -2$ that have 0 are $\{0, 1\}$ and $\{0, -1\}$.